WARPED INFINITELY DIVISIBLE CASCADES : BEYOND POWER LAWS

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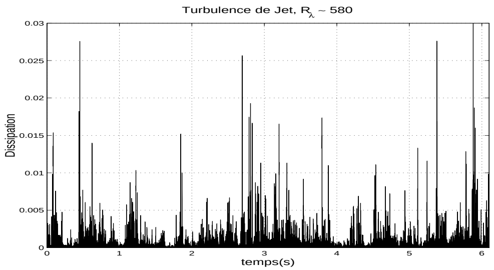
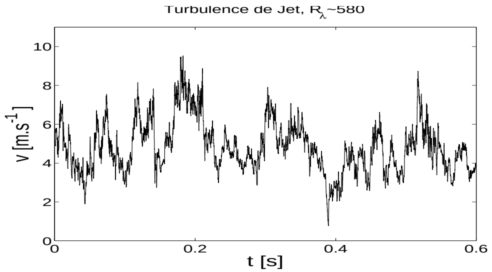
Scaling laws in turbulence



dissipation $\mathbb{E}\varepsilon_r^q \sim r^{\tau(q)}$



velocity $\mathbb{E}|\delta v_r|^q \sim r^{q/3+\tau(q/3)}$

Analysis

multifractal formalism ($\sim \tau^{\zeta(q)}$)
log-inf. div. casc. ($H(q) \cdot n(\tau)$)
multiplicative cascades
$\Rightarrow$ phenomenology
(Richardson's cascade)

Synthesis

binomial cascades,
wavelet coeff., MRW...
multiplicative cascades
$\Rightarrow$ algorithms
(binomial, wavelets...)

Infinitely divisible scaling laws

$$\mathbb{E}\,|\delta_\tau X|^q = c_q \exp[-H(q)\cdot n(\tau)]$$

- ✔ $n(\tau) = -\log\tau \implies \mathbb{E}\,|\delta_\tau X|^q = c_q \tau^{H(q)} \equiv c_q \tau^{\zeta(q)}$
  (scale invariance)

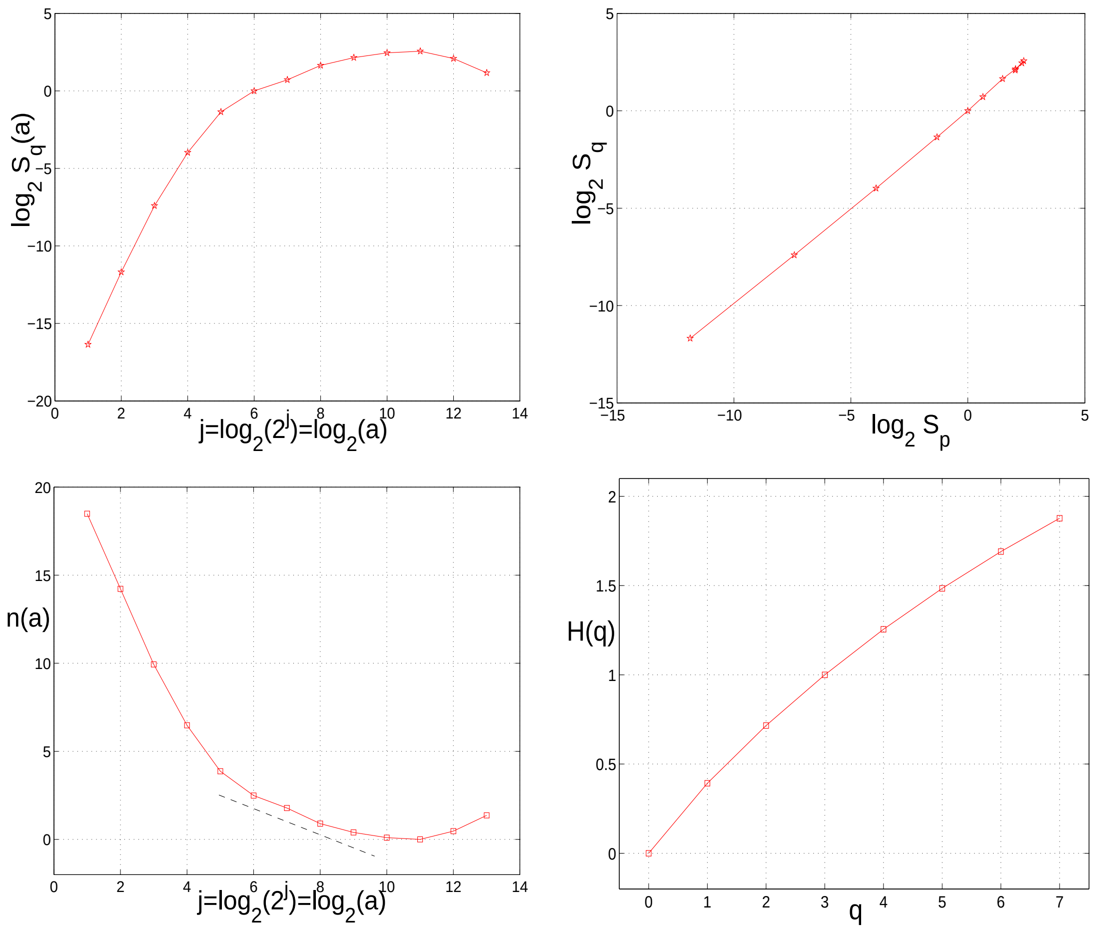
- ✔ $n(\tau) \neq -\log\tau \implies$ towards a non-scale invariant analysis :
  - turbulence `Castaing et al. 1990`
  - internet traffic `Veitch et al. 2000`

- ✔ multiplicative interpretation

How to build a signal
with infinitely divisible scaling laws ?

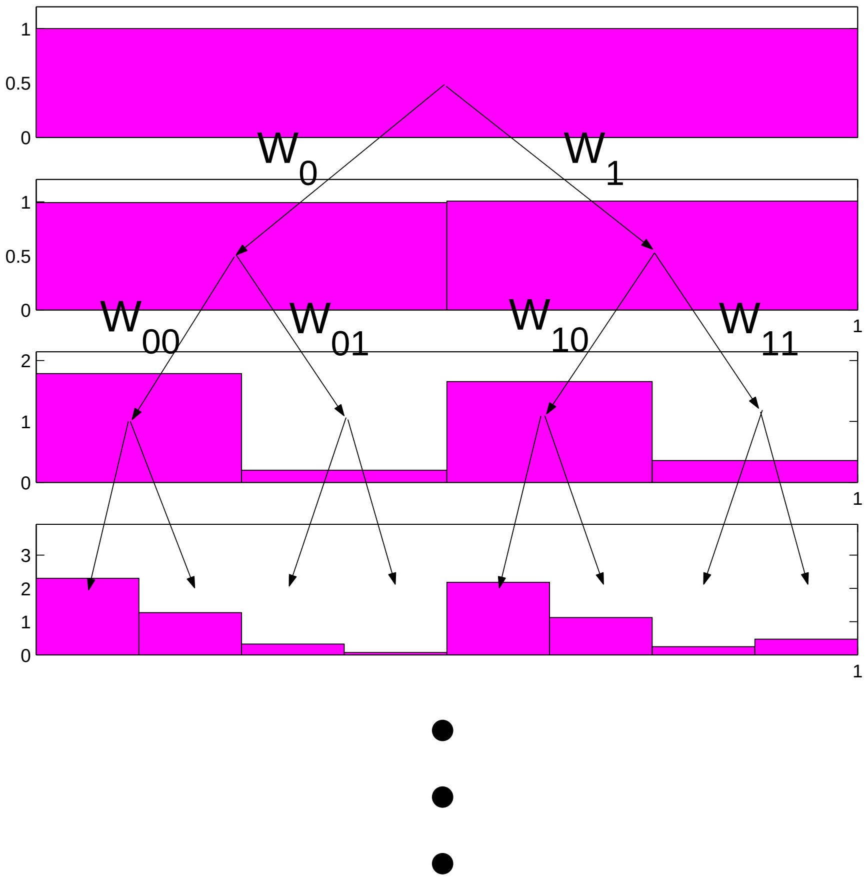
with "nice properties" : *stationary increments*, etc. . .

Example : ID scaling in turbulence



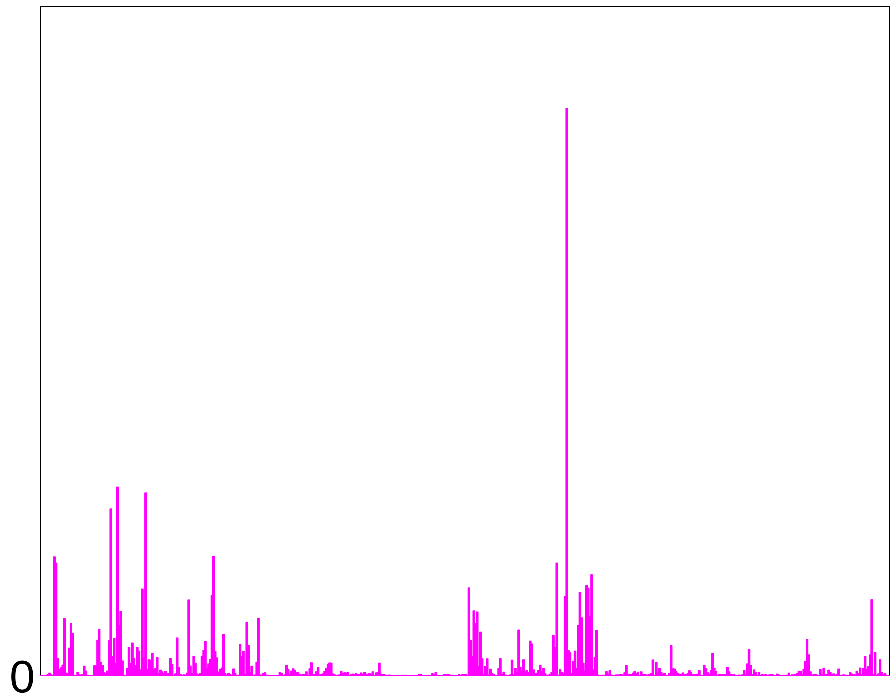
Binomial cascades

tree structure



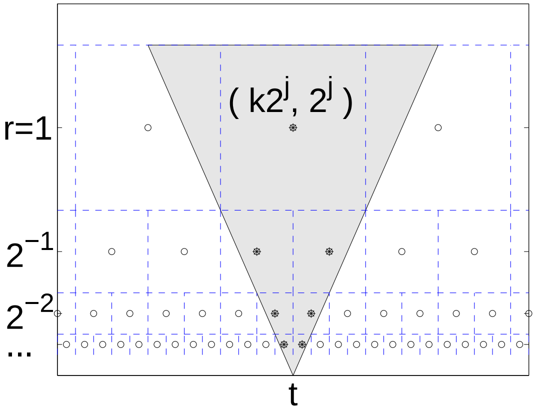
⟹

irregular signal



From binomial to infinitely divisible cascades

Binomial

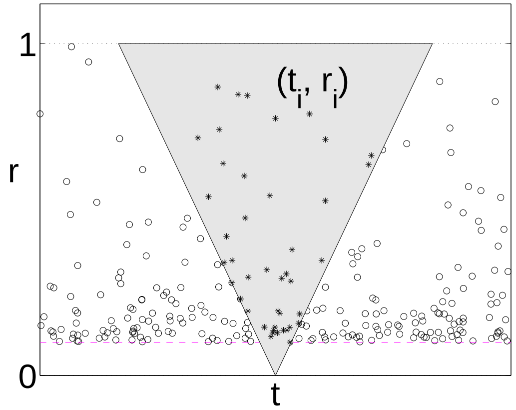


$$Q_r(t) = \prod_j \Lambda_j(t),$$

$r = 2^j$ only

`Mandelbrot 1974`

Compound Poisson



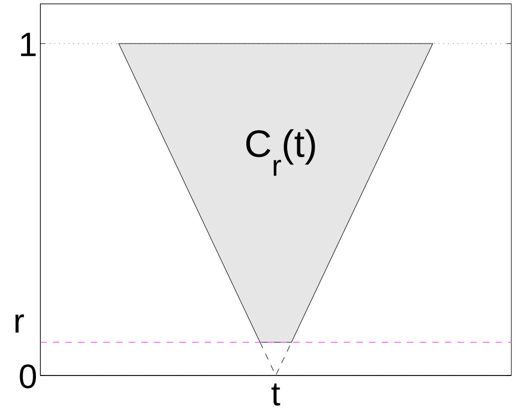
$$Q_r(t) \propto \prod_{(t_i, r_i)} W_i$$

t_i : uniform ⇒ stationary

r_i : $1/r^2$ ⇒ scaling

`Barral & Mandelbrot 2002`

Infinitely Divisible



$$Q_r(t) \propto \exp M(\mathcal{C}_r(t))$$

CONTINUOUS

MULT. CASCADE

`Schmitt & Marsan 2001`

`Muzy & Bacry 2002`

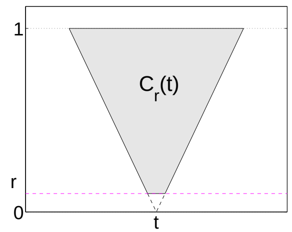
`Ch. & Riedi & Abry 2003`

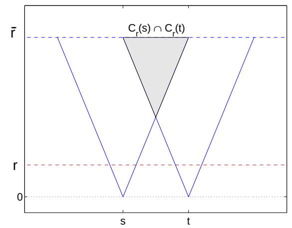
Infinitely divisible noise (IDC noise)

$\mathbf{M}(\mathcal{C}_{\mathbf{r}}(\mathbf{t}))$: moment gen. func. $= \exp\left[-m(\mathcal{C}_r)\,\rho(q)\right]$ (inv. éch. $= r^{\rho(q)}$)

- G = infinitely divisible distr., moment gen. func. $\tilde{G}(q) = e^{-\rho(q)}$,
- positive measure $dm(t,r)$ [i. é. $= \frac{(1+\delta(1-r))dt\,dr}{r^2}$],

$$Q_r(t) = \frac{\exp\left[M(\mathcal{C}_r(t))\right]}{\mathbb{E}[\exp M(\mathcal{C}_r(t))]}$$





$\Longrightarrow$ $Q_r(t)$ is stationary

$$\varphi(q) = \rho(q) - q\rho(1) \Longrightarrow \mathbb{E}[Q_r^q] = \exp\left[-\varphi(q)m(\mathcal{C}_r)\right]$$

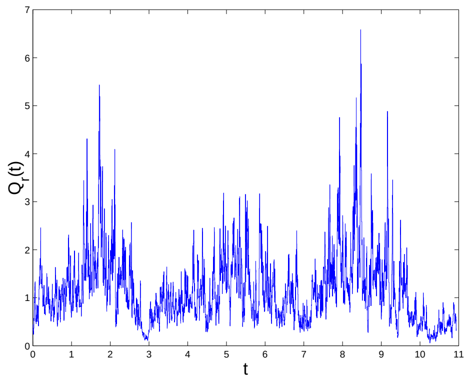
[i.é. $\mathbb{E}[Q_r^q] = e^{-\varphi(q)} \cdot r^{\varphi(q)}$]

Infinitely divisible motion (IDC motion)

Pb : Q_r degenerates when $r \longrightarrow 0\ldots$

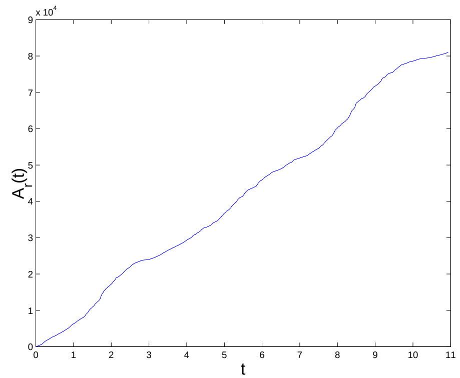
Solution : $A_r(t) = \int_0^t Q_r(s)\, ds \Longrightarrow \mathbb{E}A_r(t) = t$

Q_r



$\Longrightarrow$

A_r



$A(t) = \lim_{r \longrightarrow 0} A_r(t)$ ... has stationary increments

and $\mathbb{E}\delta_\tau A^q \sim \tau^q \exp\left[-\varphi(q) m(\mathcal{C}_\tau)\right]$

IDC cascades obey IDC scaling

locally averaged dissipation $\equiv \quad \varepsilon_{\tau}(t) = \frac{[A(t+\tau) - A(t)]}{\tau}$

[in practice : $\varepsilon_{\tau}^{(r)}(t) = \frac{1}{\tau} \int_{t}^{t+\tau} Q_r(t) dt$]

Scaling of moments $\equiv$ Inf. Div. scaling laws

$$\mathbb{E} \varepsilon_{\tau}^{q} \sim \exp\left[-\varphi(q) \cdot m(\mathcal{C}_{\tau}) \right] \equiv \exp\left[-\mathbf{H}(q) \cdot \mathbf{n}(\tau) \right]$$

$\Longrightarrow$ synthesis of a turbulent dissipation with Inf. Div. scaling

Infinitely divisible random walk (IDC random walk)

fractional **B**rownian **m**otion B_H, $A(t)$ an Inf. Div. measure,

$$V_H(t) = B_H(A(t)), \quad t \in \mathbb{R}^+$$

Rappel
$$\begin{cases} B_H \text{ has stationary increments} \\ \forall\, t \in \mathbb{R}^+, \mathbb{E}|B_H(t)|^q = t^{qH} \cdot \mathbb{E}|B_H(1)|^q, \end{cases}$$

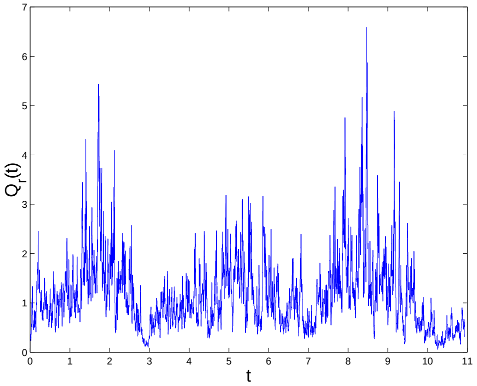
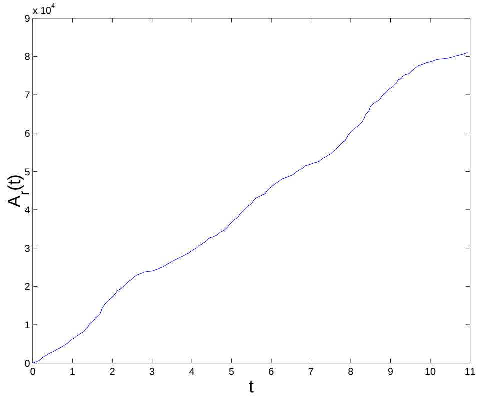
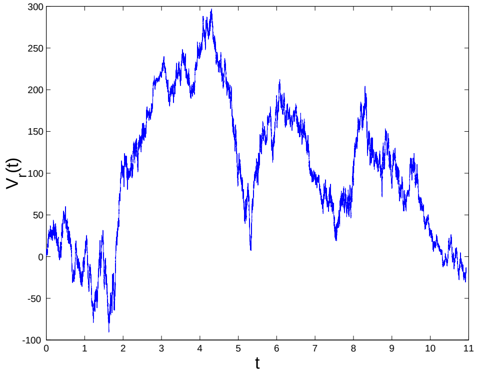
V_H has stationary increments,

positive/negative fluctuations,

and

$$\mathbb{E}|\delta_\tau V_H|^q \sim \tau^{qH} \exp\left[-\varphi(qH) m(\mathcal{C}_\tau)\right]$$

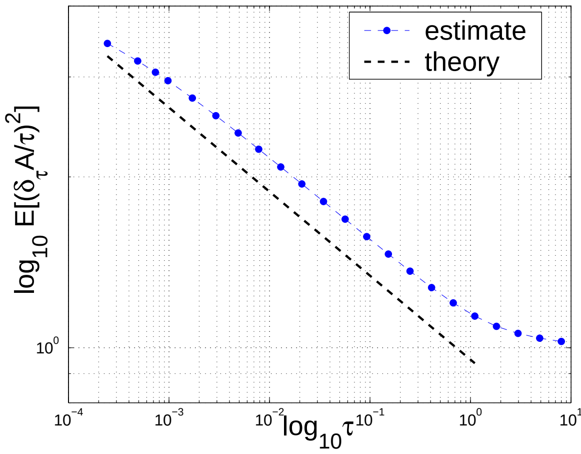
In summary...

noise (density)		**motion** (measure)		**random walk**
	$\overset{\int Q_r}{\Longrightarrow}$		$\overset{B_H(A)}{\Longrightarrow}$	
$\mathbb{E}[Q_r{}^q]$		$\mathbb{E}\delta_\tau A^q$		$\mathbb{E}\lvert\delta_\tau V_H\rvert^q$
$=$		$\aleph$		$\aleph$
$\exp[-\varphi(q) m(\mathcal{C}_r)]$		$\tau^q \exp\left[-\varphi(q) m(\mathcal{C}_\tau)\right]$		$\tau^{qH} \exp\left[-\varphi(qH) m(\mathcal{C}_\tau)\right]$

- continuous time ($t \in \mathbb{R}^+$), stationary increments, continuous scale invariance, $\forall \varphi(q)$ of an Inf. Div. distribution,
- MATLAB procedures

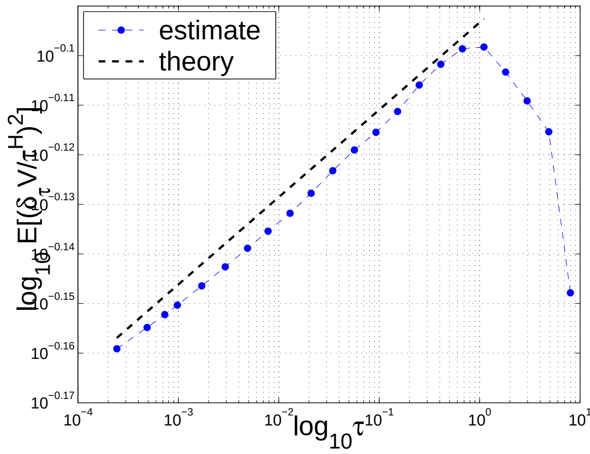
Scale invariance and power laws

$\mathbb{E}[(\delta_\tau A/\tau)^2]$



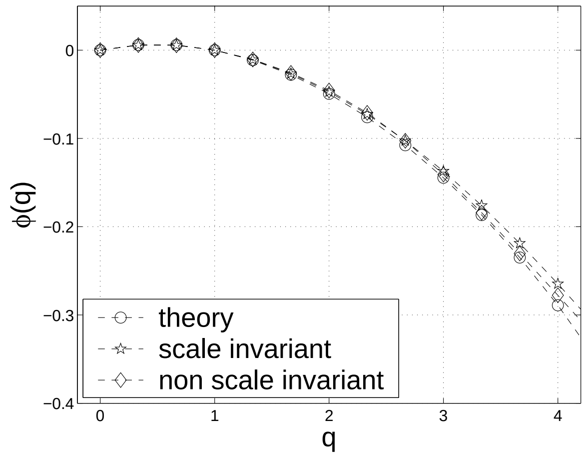
$\varphi(2)\log\tau$

$\mathbb{E}[(\delta_\tau V/\tau^H)^2]$



$\varphi(2H)\log\tau$

$\varphi(q)$



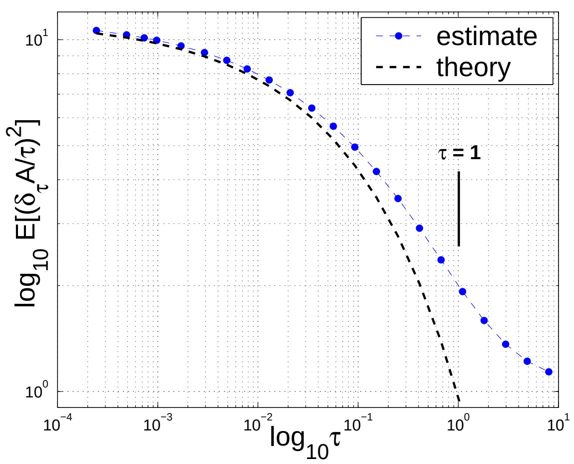
linear behaviours in log-log diagrams

$\Downarrow$

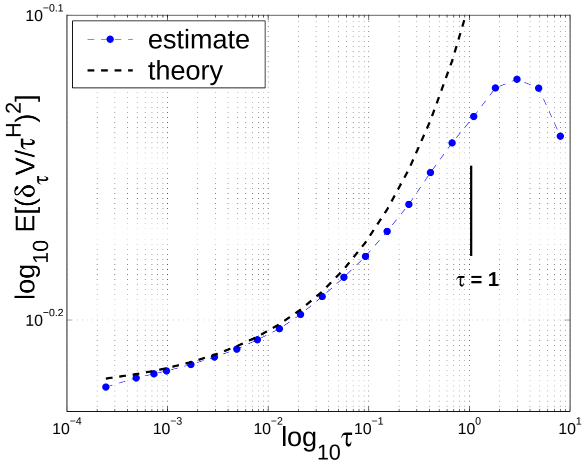
POWER LAWS

Beyond power laws...

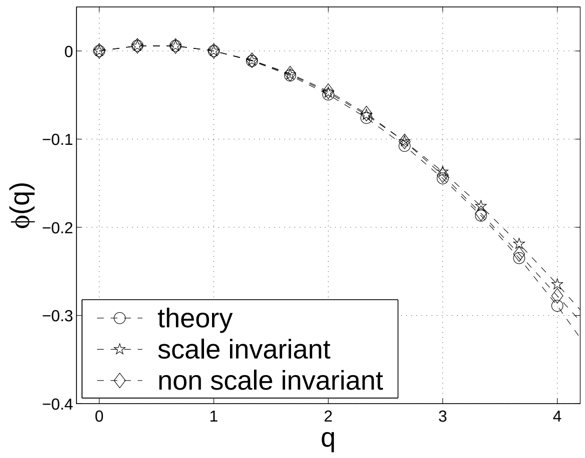
$\mathbb{E}[(\delta_\tau A/\tau)^2]$



$\mathbb{E}[(\delta_\tau V/\tau^H)^2]$



$\varphi(q)$



$-\varphi(2)m(\mathcal{C}_\tau)$ $\quad$ $-\varphi(2H)m(\mathcal{C}_\tau)$

non-linear behaviours in log-log diagrams

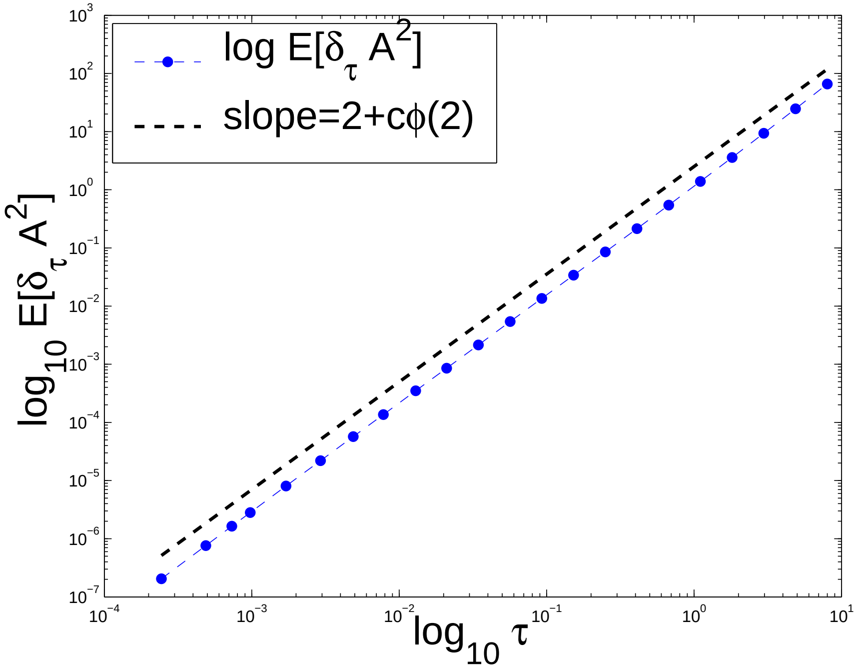
$\Downarrow$

~~POWER LAWS~~

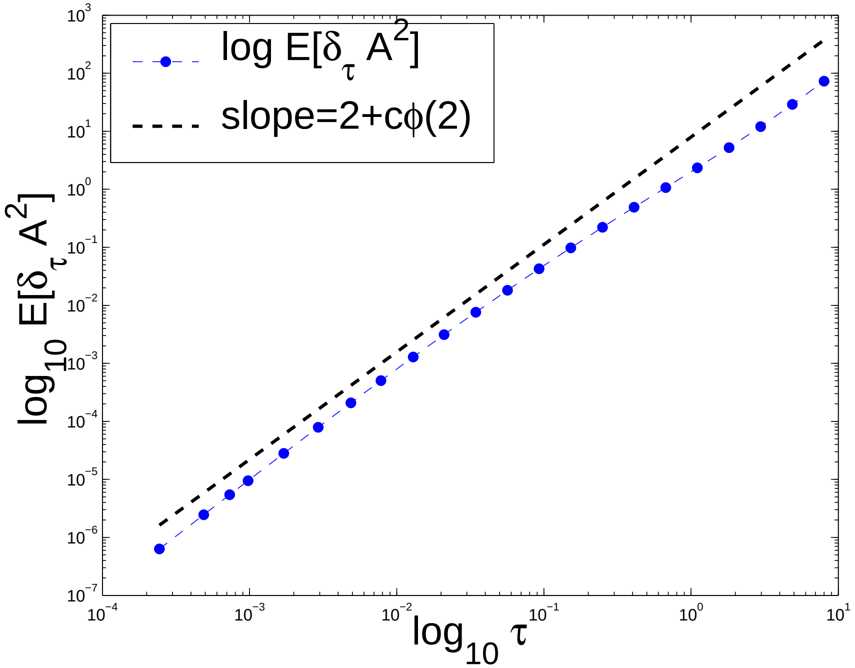
warped cascade

$\mathbf{E}\delta_\tau A^2$ multifractal / warped

MULTIFRACTAL



WARPED



A simple example

$$\boldsymbol{dm(t,r) = \frac{dt\,dr}{r^{2+\beta}}}$$

yields

$$\boldsymbol{m(\mathcal{C}_\tau)} \underset{\tau\to 0}{\sim} \begin{cases} \tau^{-\beta} \text{ if } \beta > 0, \\ -\ln\tau \text{ if } \beta = 0, \\ \boldsymbol{\frac{1-\tau^{-\beta}}{\beta}} \to \boldsymbol{C} \text{ if } \boldsymbol{\beta < 0}\ . \end{cases}$$

▼ **Multifractal Analysis** $\Longrightarrow$ $\boldsymbol{\lim_{\tau\to 0}}$

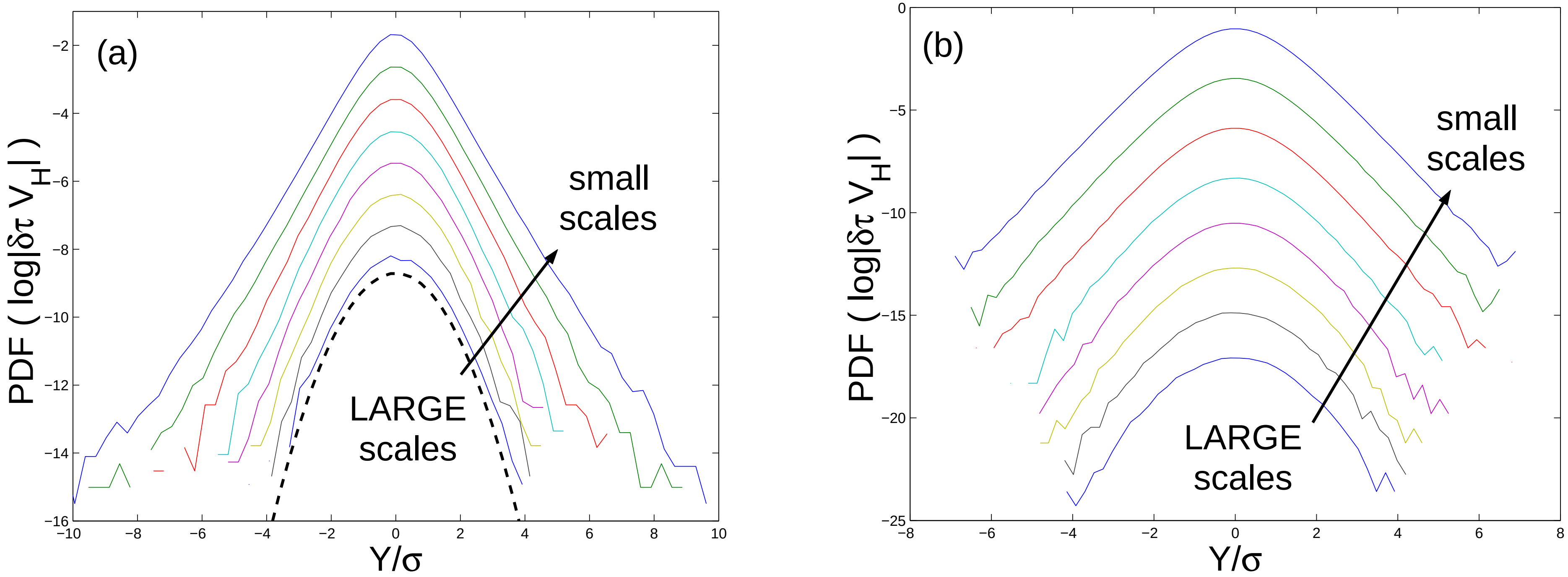
then the case $\boldsymbol{\beta < 0}$ presents no interest for multifractal analysis ($\sim \tau^q$).

▲ **LID Analysis** $\Longrightarrow$ ~~$\boldsymbol{\lim_{\tau\to 0}}$~~

interest = **evolution through the scales**

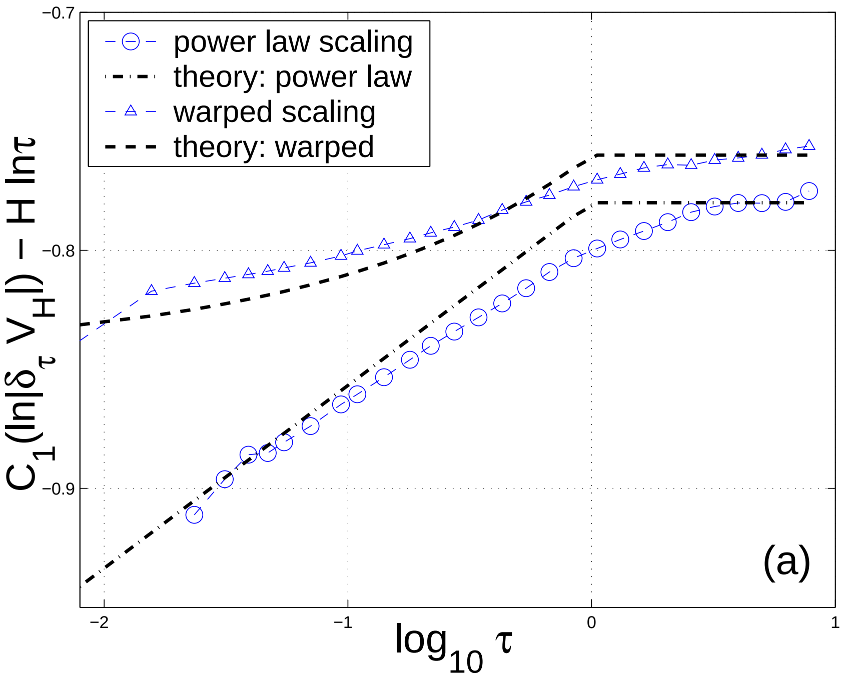
Evolution of probability density functions

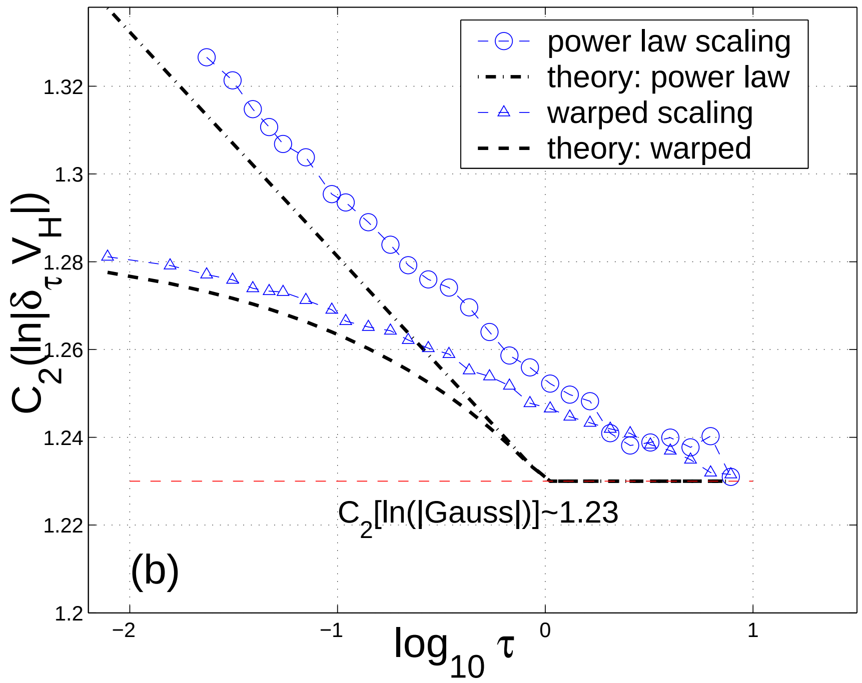
MULTIFRACTAL $\neq$ WARPED $(\beta = -0.4)$



from gaussian to *non gaussian* towards small scales
(ex : kurtosis...)

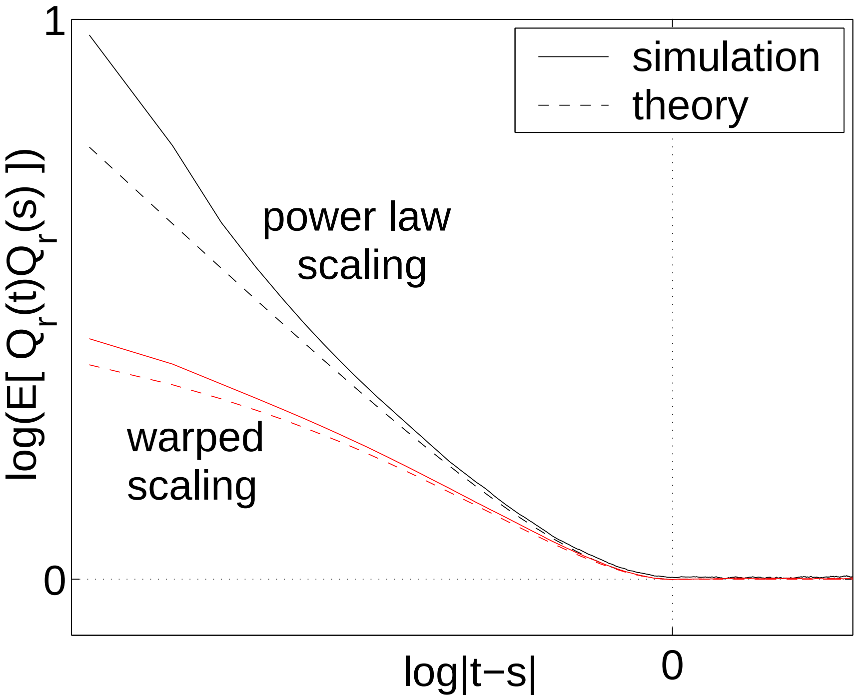
Cumulants of log(increments of $\mathbf{V_H}$)

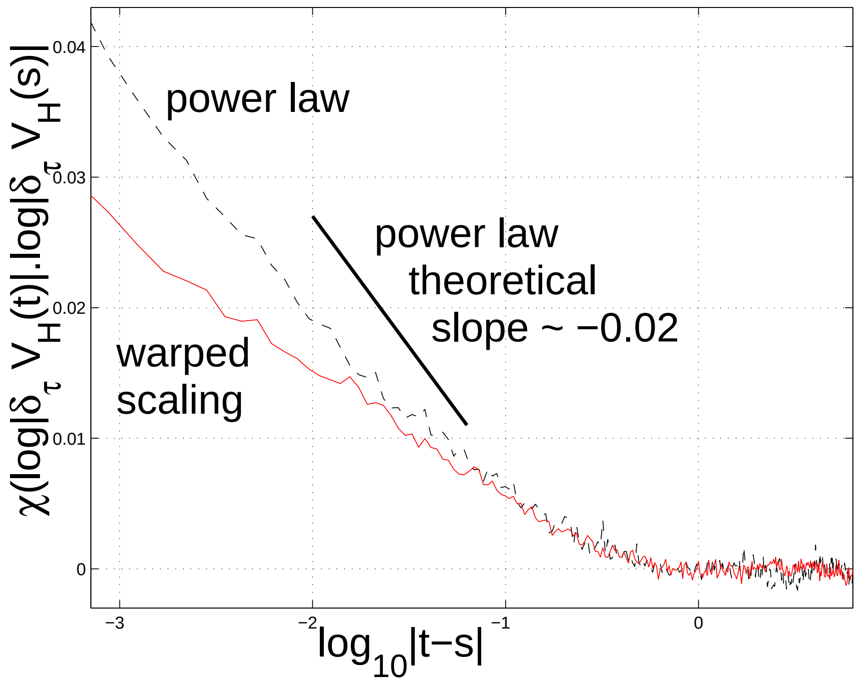




$$\text{(a)}\, C_1 - H \ln\tau \simeq -H\varphi'(0)m(\mathcal{C}_\tau) \quad \text{(b)}\, C_2 \simeq -H^2\varphi''(0)m(\mathcal{C}_\tau)$$

"Warped" autocorrelation function





$\mathbb{E}[Q_r(t)Q_r(s)]$ Autocorrelation of $\ln|\delta_\tau V_H|$

Intermediate conclusions

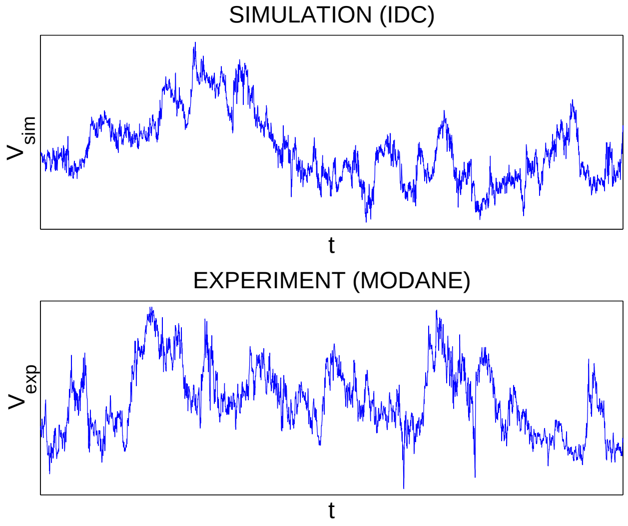
- ✔ **we can *synthesize* Inf. Div. models,**
- ✔ **warped cascades** : *a first controlled departure from power laws,*
- ✔ the effect is weak for V_H...,
- ✔ **applications :** turbulence, internet traffic, finance...

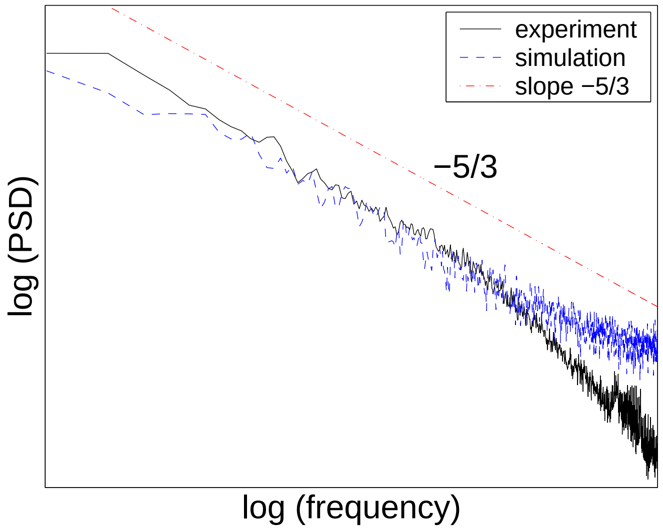
Next ?

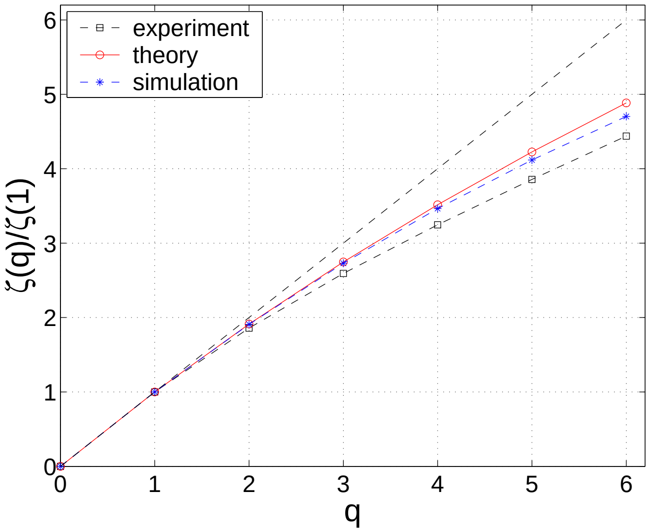
- ✔ **farther from power laws : is it possible ?. . .**
- ✔ **other ways than IDC ?** (pb of integration. . .)
- ✔ **skewness** ?
- ✔ **analysis :** estimation, prediction. . .
- ✔ **multidimensional/multivariate cascades**
- ✔ **a toolbox with synthesis algorithms.**

IDC *vs* She-Lévêque model and "Modane" turbulent signal

$\equiv$ Inf. Div. random walk V_H *versus* velocity v







V_H *versus* Modane | Spectra | Exponents $\zeta(q)$

$\Longrightarrow$ **artificial signal obeying the She-Lévêque model**

De l'auto-similarité aux Cascades Infiniment Divisibles

$$[\ \delta_\tau X(t) = X(t+\tau) - X(t)\ ]$$

☛ **AUTO-SIMILARITÉ**

$$\mathbb{E}\,|\delta_\tau X|^q = \mathbb{E}\,|X(1)|^q \cdot |\tau|^{qH}$$

ex : f.B.m., Linear Fractional Stable Motion...

☛ **FORMALISME MULTIFRACTAL**

$$\mathbb{E}\,|\delta_\tau X|^q = c_q\ \tau^{\zeta(q)}$$

ex : casc. binomiales, casc. aléa. ondelette, multifractal random walk (MRW)...

☛ **CASCADES INF. DIVISIBLES**

$$\mathbb{E}\,|\delta_\tau X|^q = c_q \exp[-H(q) \cdot n(\tau)]$$

- $H(q) \equiv \zeta(q)$ si $n(\tau) = -\log\tau$, (multifractal)
- a priori $n(\tau) \neq -\log\tau$.

ex : ... ? ? ?

A stochastic equation for the Log-Normal case

Notation : $Q_r \equiv \varepsilon_\lambda$ with $\lambda = \dfrac{1}{r} \gg 1$

$$\varepsilon_\lambda(t) = \lambda^{-\mu/2} \exp\left(\mu^{1/2} \int_{t+1-\lambda}^{t} (t+1-u)^{-1/2}\, dB(u) \right)$$

$$\Downarrow$$

$$\begin{cases} d\varepsilon_\lambda(t) = \sqrt{\mu}\, \varepsilon_\lambda(t) \left(dB(t) + \frac{1}{2}(1 - W(t))dt \right) \\ W(t) = \sqrt{\mu} \int_{t+1-\lambda}^{t} (t+1-u)^{-3/2} dB(u) \end{cases}$$

[F.G. Schmitt, Eur. Phys. J. B, 2003]