# CASCADES INFINIMENT DIVISIBLES : AU-DELÀ DES LOIS DE PUISSANCE

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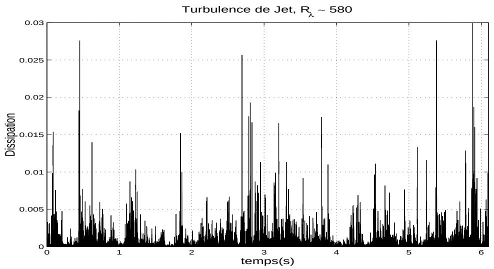
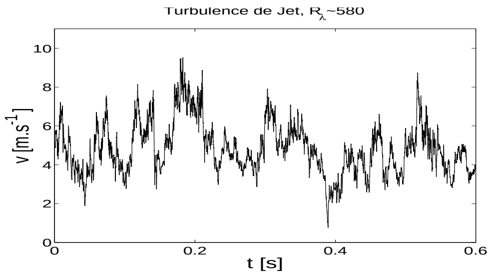
# Lois d'échelle et Turbulence




dissipation $\mathbb{E}\varepsilon_r^q \sim r^{\tau(q)}$




vitesse $\mathbb{E}|\delta v_r|^q \sim r^{q/3+\tau(q/3)}$

## Analyse

formalisme multifractal ($\sim \tau^{\zeta(q)}$)
casc. log-inf. div. ($H(q) \cdot n(\tau)$)
cascades multiplicatives
$\Rightarrow$ phénoménologie
(cascade de Richardson)

## Simulation

cascades binomiales,
coeff. ondelette, MRW...
cascades multiplicatives
$\Rightarrow$ algorithmes
(binomiale, ondelettes...)

# Cascades Log-Infiniment Divisibles (CLID)

$$\mathbb{E}\,|\delta_\tau X|^q = c_q \exp[-H(q)\cdot n(\tau)]$$

- ✔ $n(\tau) = -\log\tau \implies \mathbb{E}\,|\delta_\tau X|^q = c_q \tau^{H(q)} \equiv c_q \tau^{\zeta(q)}$
  (invariance d'échelle)

- ✔ $n(\tau) \neq -\log\tau \implies$ vers une analyse non-invariante d'échelle :
  - turbulence `Castaing et al. 1990`
  - trafic internet `Veitch et al. 2000`

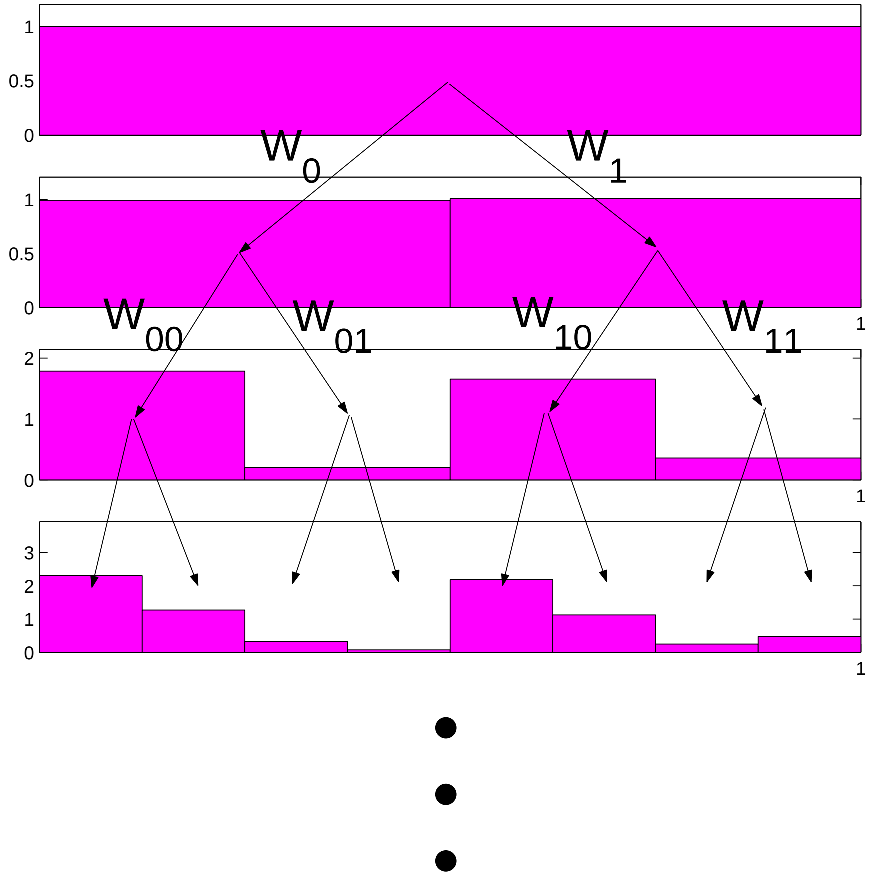
- ✔ interprétation multiplicative

**Comment construire un signal respectant une CLID ?**

possédant de "bonnes propriétés" : *accroissements stationnaires*, etc. . .

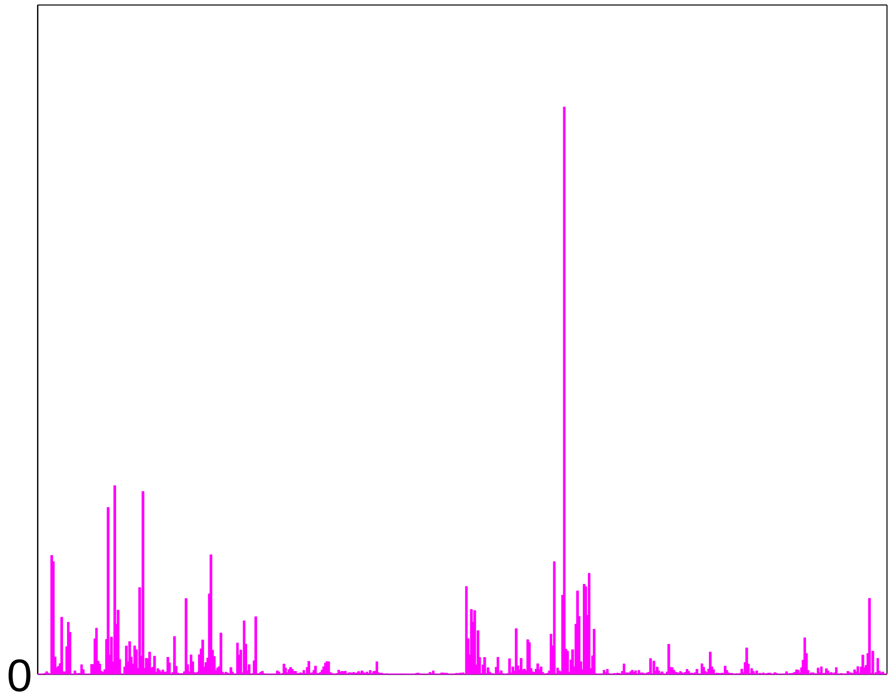
# Principe des cascades binomiales

structure arborescente



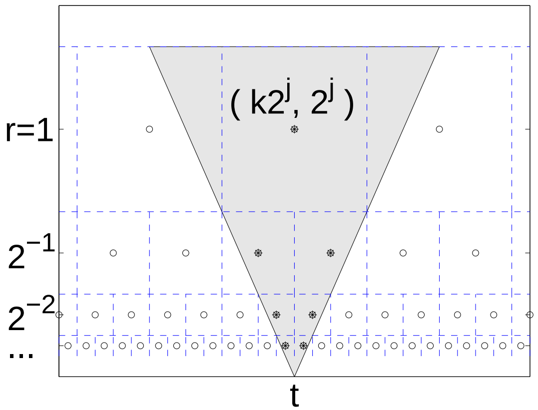

$\Longrightarrow$

signal irrégulier



# Des cascades binomiales aux cascades infiniment divisibles

## binomiale

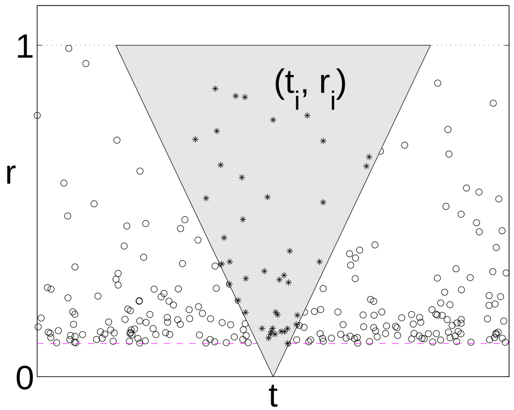



$$Q_r(t) = \prod_j \Lambda_j(t),$$

$r = 2^j$ seulement

Mandelbrot 1974

## Poisson composée



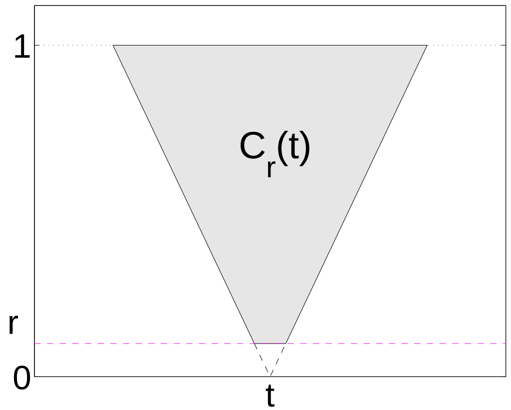

$$Q_r(t) \propto \prod_{(t_i, r_i)} W_i$$

$t_i$ : uniforme $\Rightarrow$ stationnaire

$r_i$ : $1/r^2$ $\Rightarrow$ scaling

Barral & Mandelbrot 2002

## Infiniment Divisible




$$Q_r(t) \propto \exp M(\mathcal{C}_r(t))$$

CASCADE MULT. CONTINUE

Muzy & Bacry 2002

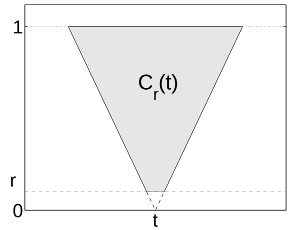
Ch. & Riedi & Abry 2003

# Bruit log-infiniment divisible (IDC noise)

$\mathbf{M}(\mathcal{C}_\mathbf{r}(\mathbf{t}))$ : fct. génér. moments $= \exp\left[-m(\mathcal{C}_r)\,\rho(q)\right]$ (inv. éch. $= r^{\rho(q)}$)

- $G$ = distr. infiniment divisible, fct. génér. moments $\tilde{G}(q) = e^{-\rho(q)}$,
- mesure positive sur plan temps-échelle $dm(t,r)$ (inv. éch. $= \frac{c\,dt\,dr}{r^2}$),

$$Q_r(t) = \frac{\exp\left[M(\mathcal{C}_r(t))\right]}{\mathbb{E}[\exp M(\mathcal{C}_r(t))]}$$




$\Longrightarrow$ $Q_r(t)$ est stationnaire

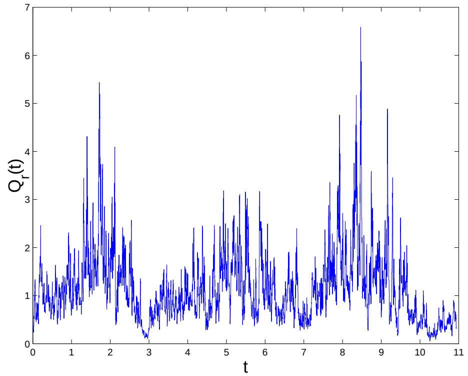
$$\varphi(q) = \rho(q) - q\rho(1) \Longrightarrow \mathbb{E}[Q_r^q] = \exp\left[-\varphi(q)m(\mathcal{C}_r)\right]$$

# Mouvement log-infiniment divisible (IDC motion)

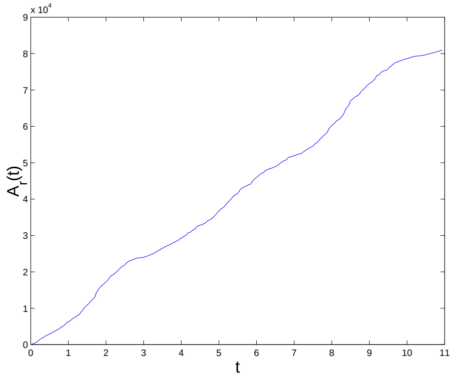
**Pb :** $Q_r$ dégénère lorsque $r \longrightarrow 0\ldots$

**Solution :** $A_r(t) = \int_0^t Q_r(s)\, ds \Longrightarrow \mathbb{E} A_r(t) = t$

$Q_r$ $A_r$




$\Longrightarrow$




$A(t) = \lim_{r \to 0} A_r(t)$ ... est à accroissements stationnaires

et $\mathbb{E} \delta_\tau A^q \sim \tau^q \exp\left[-\varphi(q) m(\mathcal{C}_\tau)\right]$

# Marche aléatoire infiniment divisible (IDC random walk)

**m**ouvement **B**rownien **f**ractionnaire $B_H$, $A(t)$ une mesure Log. Inf. Div.,

$$V_H(t) = B_H(A(t)), \quad t \in \mathbb{R}^+$$

Rappel
$$\begin{cases} B_H \text{ est à accroissements stationnaires} \\ \forall\, t \in \mathbb{R}^+, \mathsf{E}|B_H(t)|^q = t^{qH} \cdot \mathsf{E}|B_H(1)|^q, \end{cases}$$

$V_H$ est à accroissements stationnaires,

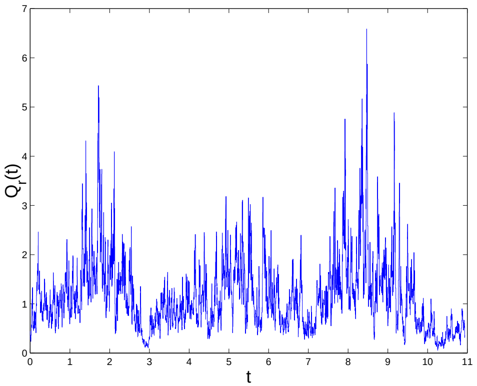
fluctuations positives/négatives,

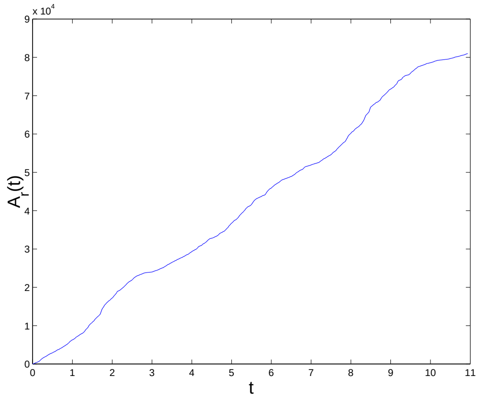
et

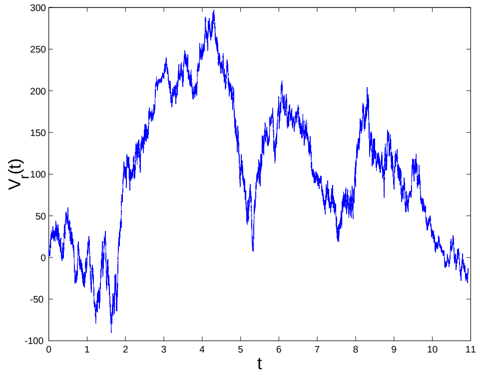
$$\mathsf{E}|\delta_\tau V_H|^q \sim \tau^{qH} \exp\left[-\varphi(qH) m(\mathcal{C}_\tau)\right]$$

# En résumé...

bruit (densité) $\xrightarrow{\int Q_r}$ mouvement (mesure) $\xrightarrow{B_H(A)}$ marche aléatoire




$$\mathbb{E}[Q_r{}^q] = \exp[-\varphi(q) m(\mathcal{C}_r)]$$

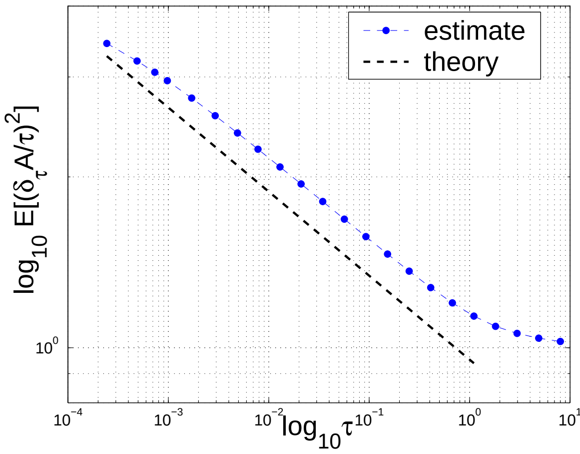
$$\mathbb{E}\delta_\tau A^q \aleph \tau^q \exp\left[-\varphi(q) m(\mathcal{C}_\tau)\right]$$

$$\mathbb{E}|\delta_\tau V_H|^q \aleph \tau^{qH} \exp\left[-\varphi(qH) m(\mathcal{C}_\tau)\right]$$

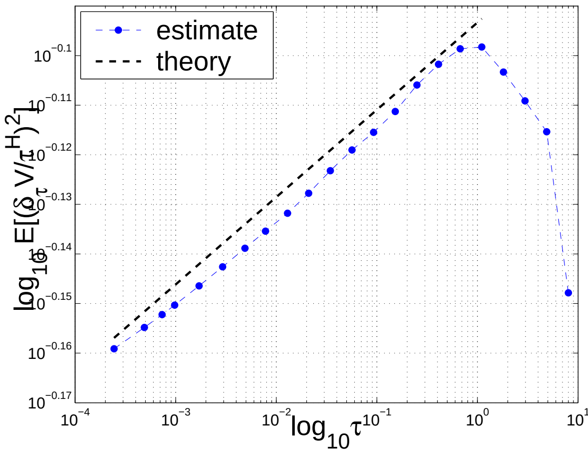
- temps continu ($t \in \mathbb{R}^+$), accroissements stationnaires, invariance d'échelle continue, $\forall \varphi(q)$ d'une distribution Inf. Div.,
- procédures MATLAB

# Invariance d’échelle et lois de puissance

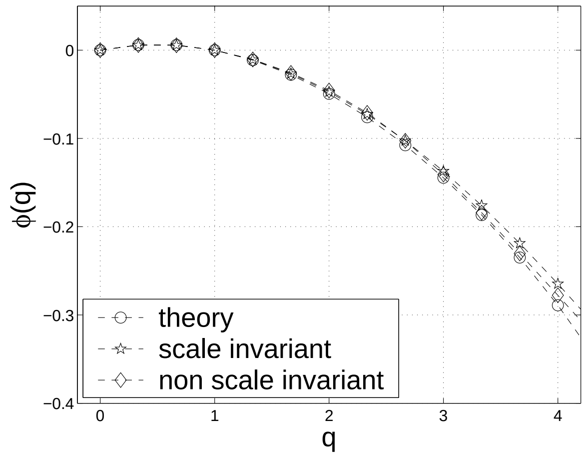
$\mathbb{E}[(\delta_\tau A/\tau)^2]$




$\mathbb{E}[(\delta_\tau V/\tau^H)^2]$




$\varphi(q)$




$\varphi(2)\log\tau$ $\varphi(2H)\log\tau$

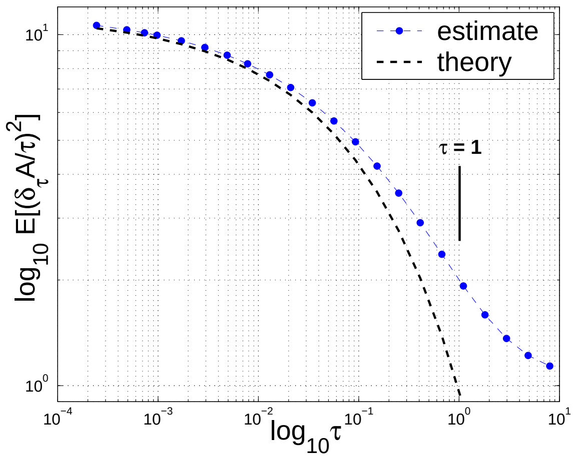
comportements linéaires dans diagrammes log-log

$\Downarrow$

Lois de Puissance

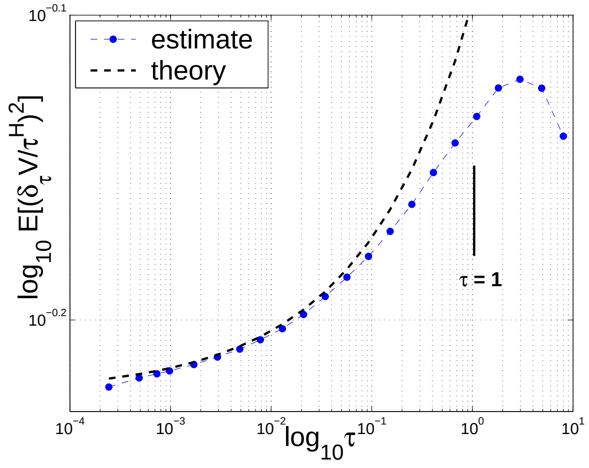
# Au-delà des lois de puissance...

$\mathbb{E}[(\delta_\tau A/\tau)^2]$



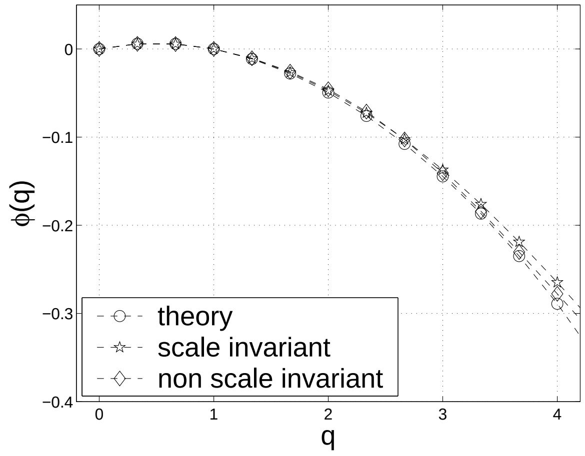

$-\varphi(2)m(\mathcal{C}_\tau)$

$\mathbb{E}[(\delta_\tau V/\tau^H)^2]$




$-\varphi(2H)m(\mathcal{C}_\tau)$

$\varphi(q)$




comportements non-linéaires dans diagrammes log-log

$\Downarrow$

~~Lois de Puissance~~

# Conclusions & Perspectives

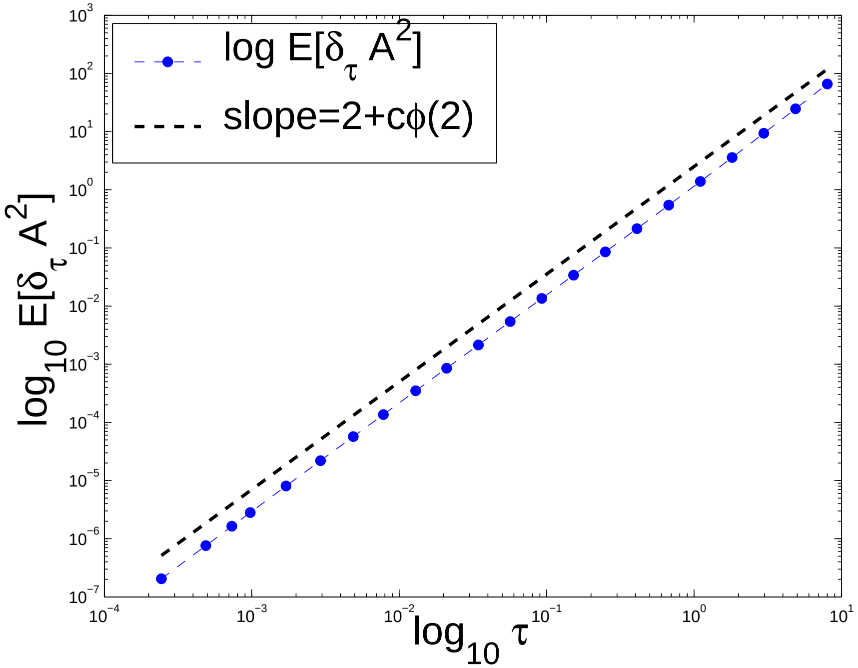
- ✔ **modèles Log-Inf. Div. simulables !**
- ✔ **vers des cascades non invariantes d'échelle...**
- ✔ **analyse :** test des outils (estimation, prédiction...),
- ✔ **turbulence**, **trafic internet**, **finance**, **nuages**...
- ✔ CLID **multidimensionnelles** ?
- ✔ **skewness** ?
- ✔ existe-t-il **d'autres multifractals que les CLID** ?

**Références**

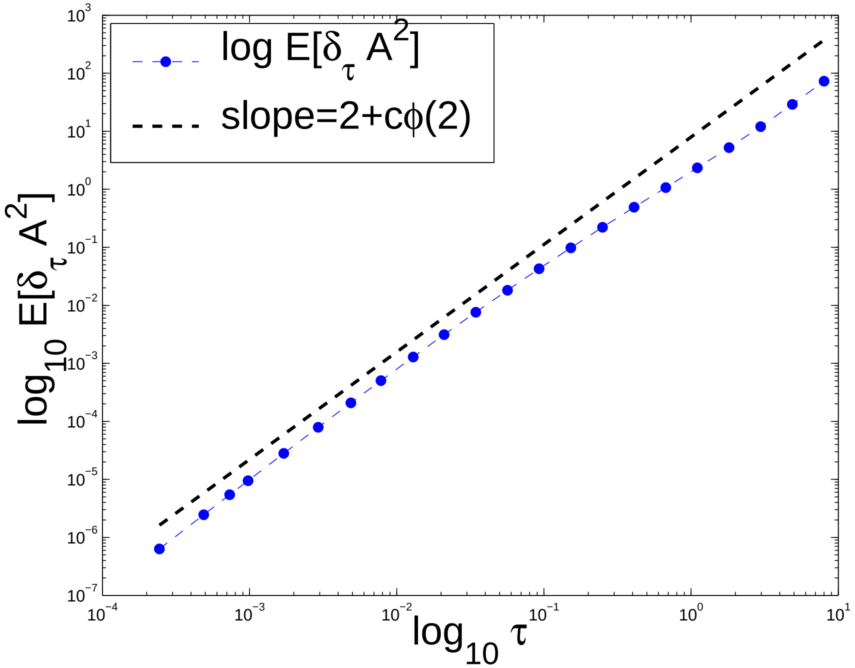
- *On non scale invariant infinitely divisible cascades*, P. Chainais, R. Riedi, P. Abry.,*preprint*.
- *Scale invariant infinitely divisible cascades*, P. Chainais, R. Riedi, P. Abry., *Proc. of PSIP'2003*.
- *Compound Poisson Cascades*, P. Chainais, R. Riedi, P. Abry, *Conf. Self-Similarity & Applications* 2002.
- *Multifractal stationary random measures and multifractal random walks with log-infinitely divisible scaling laws*, J.-F. Muzy, E. Bacry, *Phys. Rev. E*, 2002.

# $\mathbf{E}\delta_\tau A^2$ inv. d'éch. / non inv. d'éch.

INVARIANT D'ÉCHELLE

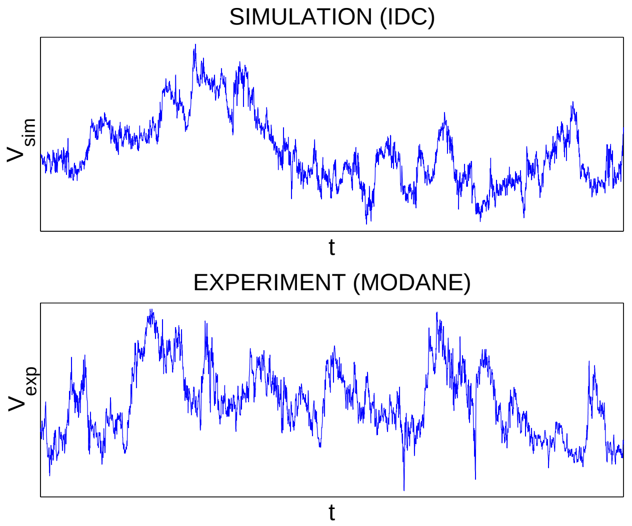



NON-INVARIANT D'ÉCHELLE

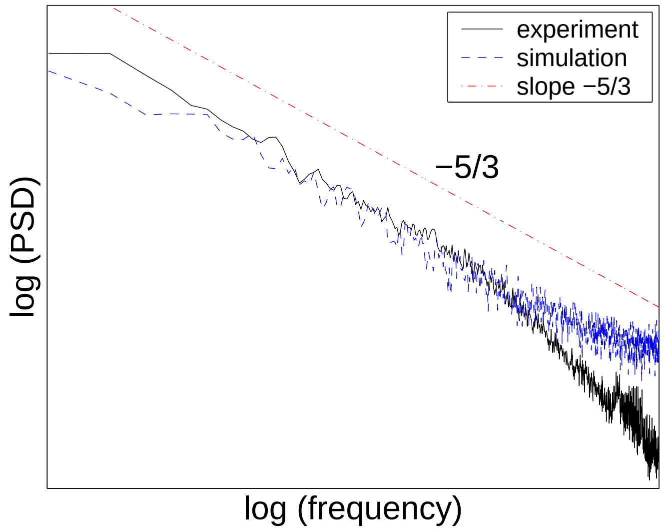


# CLID *vs* modèle She-Lévêque et signal "Modane"

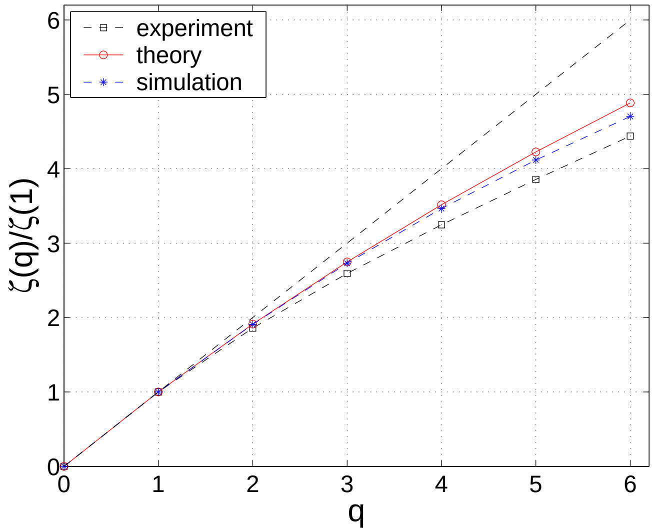
$\equiv$ marche aléatoire Log-Inf. Div. $V_H$ *versus* vitesse $v$




$V_H$ *versus* Modane




Spectres




Exposants $\zeta(q)$

$\Longrightarrow$ signal artificiel respectant le modèle de She-Lévêque