MULTIFRACTAL ANALYSIS AND α-STABLE PROCESSES: A METHODOLOGICAL CONTRIBUTION

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ABSTRACT

This work is a contribution to the analysis of the procedure, based on wavelet coefficient partition functions, commonly used to estimate the Legendre multifractal spectrum. The procedure is applied to two examples, a fractional Brownian motion in multifractal time and a self-similar α-stable process, whose sample paths exhibit irregularities that *by eye* appear very close. We observe that, for the second example, this analysis results in a qualitatively inaccurate estimation of its multifractal spectrum, and a related masking of the α-stable nature of the process. We explain the origin of this error through a detailed analysis of the partition functions of the self-similar α-stable process. Such a study is made possible by the specific properties of the wavelet coefficients of such processes. We indicate how the estimation procedure might be modified to avoid such errors.

1. INTRODUCTION

Signals which present both strong variability on many time scales, as well as highly irregular local structure, appear in many contexts. *Multifractal* analysis [13] has become one of the most widely known tools in the study of such signals. Multifractals (MF) allow the compact description of complex forms of *scaling behaviour* which go well beyond that of more traditional scaling models such as the *fractional Brownian motion* (fBm). Another central feature is their strongly non-Gaussian character. Such models therefore imply non-redundant scaling behaviour of moments of many orders. These are commonly studied through the *structure* or *partition functions* $E|X(t+\tau)-X(t)|^q$, based on the increments of the process X, which for multifractal processes are expected to obey power laws

$$\mathbb{E}|X(t+\tau)-X(t)|^q \sim \tau^{\zeta(q)} \text{ for } \tau \to 0.$$

The $\zeta(q)$ are often estimated through linear regression in log-log diagrams. From these exponents information on the *Legendre multifractal spectrum* $D(h)$ (see below) can be extracted, which describes the relative abundance of local scaling exponents h of different orders (more refined forms of MF spectra exist which are not considered here [13]).

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There is another class of models which combine in a natural way both scaling behaviour and extreme local irregularity, namely *self-similar α-stable* (α-SSS) processes [14]. Such processes are multifractal [7, 13], however they form a distinct class since their statistics of order $q \geq \alpha$ are not finite (unless $\alpha = 2$, the Gaussian case). This is a source of irregularity which is different in nature to that of other kinds of multifractals, for example the *multiplicative cascades* introduced by Mandelbrot, which have lognormal marginals for which all moments are finite. We show here that despite this dramatic difference in nature, it is possible to confuse signals from these two classes when performing multifractal analysis based on the estimated $\zeta(q)$. Such an error can have powerful implications. For example in queueing analysis of telecommunications switches with multifractal input, infinite moments in the input process can engender infinite moments in the queueing process, corresponding to rather long waiting times. From two examples, a *fractional Brownian motion in multifractal time* and the *linear fractional stable motion*, a simple α-SSS process, we show that the seeds of the difficulty are twofold. First, surprisingly, there is in fact no sharp discontinuity in the estimates of the exponents $\zeta(q)$ in the α-stable case at the boundary $q = \alpha$. Second, estimation difficulties about this point can hide the differences in behaviour that do in fact exist. The main implications are that the MF spectrum is very poorly estimated, and the possible α-stable nature of the process is hidden. Through a thorough analysis of the wavelet-based partition functions of the α-stable process, we explain why these problems arise, indicate the actual behaviour of the $\zeta(q)$ estimates and the corresponding MF spectrum, and finally suggest how the errors might be avoided in practice.

2. MULTIFRACTAL ANALYSIS

It has been proposed for some time [3, 4] and proved more recently [13] that it is possible to replace an increment based approach by defining *wavelet-based structure functions* capable of capturing the Legendre MF spectrum. A wavelet approach is desirable from many statistical points of view. For example, non stationary processes with stationary increments can be rendered stationary in the wavelet domain, enabling estimation via time averages. Strong correlations in the time domain can also be dramatically reduced simply by selecting appropriately the number of *vanishing moments* of the wavelet, leading to estimates with small variance [1].

The discrete wavelet transform of a process X consists of the set of coefficients

$$d_X(j,k) = \langle \psi_{j,k}, X \rangle$$

where $\langle , \rangle$ denotes a scalar product and $\psi_{j,k}$ is a dilated and translated copy of the mother wavelet ψ [5] : $\psi_{j,k}(t) = 2^{-j}\psi(2^{-j}t - k)$. Note that for each fixed j the coefficents $\{d_X(j,k)\}$ form a stationary process. The wavelet based structure functions at scale 2^j are defined as $E|d_X(j,\cdot)|^q$, which assuming a classical multifractal formalism take the form $E|d_X(j,\cdot)|^q \sim 2^{j\zeta(q)}$. The Legendre multifractal spectrum $D(h)$, where h is the local regularity exponent, then follows from the Legendre transform of the $\zeta(q)$:

$$D(h) = \min_q (qh - \zeta(q) + 1)$$

for those h for which $D(h) > 0$ [13] (we ignore other cases).

The key practical step is the estimation of the $\zeta(q)$, which is effectively and commonly achieved via

$$S_q(j) = \frac{1}{n_j} \sum_{k=1}^{n_j} |d_X(j,k)|^q,$$

where n_j is the number of coefficients at octave j. The resulting estimation procedure consists of three steps. First, check for straight lines in the $\log_2(S_q(j))$ against $\log_2(2^j) = j$ diagram. We will refer to such plots as Logscale Diagrams (LD's). Second, estimate $\zeta(q)$ through linear regression as the slope in those diagrams. Confidence intervals can be calculated for each $\log_2(S_q(j))$, which is essential for verifying the existence of the linear fit, and for determining confidence intervals for $\zeta(q)$ estimates. Finally, an estimate of the MF spectrum is obtained as the Legendre transform of the estimated $\{\zeta(q)\}$.

3. TWO MULTIFRACTAL PROCESSES

In the next section we apply the estimation procedure described above to two very different multifractal processes: a fractional Brownian motion in multifractal time (MfBm), and an α-stable self-similar process (α-SSS). Sample paths of those processes, together with their increments, are plotted in the top row of figures 1 and 2 respectively. The parameters of each process have been chosen such that their sample paths are superficially very alike, with a similar degree of irregularity. Thus if one did not know the origin of the data, but wished to measure their multifractal spectra, one would be naturally led to perform the very same analysis procedure on each. In this section we describe the theoretical properties of these examples, and in particular their spectra.

The fractional Brownian motion in multifractal time $\mathcal{B}(t)$ [11, 13] consists of a fractional Brownian motion with self-similarity parameter H, $B_H(t)$ [14], whose time has been reparameterised: $t \to \mathcal{M}(t)$. This warping is highly unsmooth, in fact *multifractal* in the sense that $\mathcal{M}(t)$ is the distribution function of the multifractal measure characterising the limit of a stochastic binomial multiplicative cascade. We briefly recall that a binomial cascade is an iterative re-distribution of an originally uniform mass on the interval, where line segments are repeatedly divided in two, the masses being re-distributed via weights obtained by multiplying the original weights by *multipliers* resulting from independent trials of a given random variable. For a thorough review of multiplicative cascades and the MfBm, see [13]. The MF spectrum $D^{\mathcal{B}}(h)$ of the MfBm is related to that of the cascade through $D^{\mathcal{B}}(h) = D^{\mathcal{M}}(h/H)$, and the spectrum of many simple cascades can be calculated analytically. It has been shown in [10] that the exponents ζ_q of the wavelet-based structure functions of the MfBm are related to its MF spectrum.

We next consider an α-SSS process with $\alpha \in (0,2)$, defined through the integral representation [14]:

$$X(t) = \int f(t,u) M(du), \tag{1}$$

where $M(du)$ is a symmetric α-stable measure (with scale parameter σ), and $f(t,u)$ an integration kernel that controls the time dependence of the statistical properties of X. Provided that f is a well-chosen function, X is a self-similar process with self-similarity parameter H, i.e., $\forall c > 0, \{c^{-H}X(ct), t \in \mathcal{R}\} \stackrel{d}{=} \{X(t), t \in \mathcal{R}\}$, where $\stackrel{d}{=}$ denotes equality of all finite dimensional distributions. The Lévy process, $f(t,u) = \mathbf{1}(t-u \geq 0) - \mathbf{1}(u \leq 0)$, is the simplest example, as it possesses stationary and independents increments, with H simply $H = 1/\alpha$. The linear fractional stable motion (LFSM) [14], for which $f(t,u) = (t-u)_+^{(H-1/\alpha)} - (-u)_+^{(H-1/\alpha)}$, where $(u)_+ = u$ if $u \geq 0$ and 0 elsewhere, and $\alpha \in (0,2)$, $H \in (0,1)$ and $H \neq 1/\alpha$, is a stable equivalent of the fBm with stationary but dependent increments. It has been shown [6, 12] that, under mild conditions on the mother-wavelet ψ, the wavelet coefficients $d_X(j,k)$ of stable processes exist and are α-stable random variables with index $\alpha_j = \alpha$ for all j. The assumption of self similarity then implies that their scale parameters satisfy [2]

$$\sigma_j = \sigma_0 2^{jH}, \tag{2}$$

where σ_0 depends on both ψ and the nature of $f(t,u)$, and furthermore [2] that

$$\mathbb{E}|d_X(j,k)|^q = c_q 2^{jqH}, \ -1 < q < \alpha. \tag{3}$$

Absolute moments outside of $q \in (-1, \alpha)$ are infinite, and so, crucially, the exponents $\zeta_q = qH$ are likewise only defined within this range. A straightforward computation of the corresponding Legendre transform yields the tent-like multifractal spectrum shown in figure 1(f), where the two slopes are given by α over $h \in (-\infty, H)$ and -1 over $h \in [H, \infty)$ (see [7, 13] for a theoretical analysis of the MF properties of α-SSS processes).

We arrive at the central difficulty, namely that in practice one does not know if multifractal data is α-stable, and in any case α will be generally unknown, indeed a key parameter to measure. Thus it will not be known when or if the ζ_q become undefined, and the question immediately arises of what will happen if we unwittingly attempt to apply the estimation procedure described above at orders q exceeding α. To explore this question consider the following arguments. Let X be $S_\alpha(\sigma, \beta)$, that is an α-stable random variable with scale parameter σ and symmetry parameter

β, and let $F(x)$ denote its cumulative distribution function (CDF). From theorem 3, pp. 547 in [8], we have

$$\begin{cases} x^\alpha [1 - F(x)] & \xrightarrow[x\to+\infty]{} \; L(\alpha)\sigma^\alpha(1+\beta) \\ x^\alpha F(-x) & \xrightarrow[x\to+\infty]{} \; L(\alpha)\sigma^\alpha(1-\beta). \end{cases} \tag{4}$$

where

$$L(\alpha) = \begin{cases} \frac{2-\alpha}{\pi}\sin(\frac{\pi\alpha}{2})\Gamma(\alpha) & \text{if } 0 < \alpha \le 1, \\ \frac{2-\alpha}{\alpha\pi}\sin(\frac{\pi\alpha}{2})\Gamma(\alpha) & \text{if } 1 \le \alpha < 2. \end{cases} \tag{5}$$

Let $G(y)$ denote the CDF of $Y = |X|^q$ with $q > \alpha$. It follows immediately from (4) that:

$$G(y) = \begin{cases} 0 & \text{for } y \le 0, \\ 1 - [2L(\alpha)\sigma^\alpha + \mu_2(y)]\,\dfrac{1}{y^{\alpha/q}} & \text{for } y > 0, \end{cases} \tag{6}$$

where $\lim_{y\to+\infty}\mu_2(y) = 0$. The tails of $G(y)$ therefore satisfy the conditions of Theorem 5, pp. 181 in [9], proving that $G(y)$ belongs to the domain of normal attraction of a stable law $S_{\alpha/q}((2L(\alpha)\sigma^\alpha)^{q/\alpha}, 1)$. Let Z be distributed according to this law, and let $\{X_k, k = 1, \ldots, n\}$ denote i.i.d. random variables distributed as $S_\alpha(\sigma, \beta)$. It follows that

$$Z \stackrel{d}{=} \lim_{n\to+\infty} \frac{1}{n^{q/\alpha}} \sum_{k=1}^{n} |X_k|^q$$

as $\alpha/q < 1$ means that centering constants are not required. From standard results [8], we derive that (c_ℓ is the Euler constant):

$$\begin{aligned} E[\log_2|Z|] =& \frac{c_\ell}{\ln 2}\left(\frac{q}{\alpha} - 1\right) + \log_2((2L(\alpha)\sigma^\alpha)^{q/\alpha} \\ &+ \frac{q^2}{\alpha^2}\log_2\left(1 + \tan^2\left(\frac{\pi q}{2\alpha}\right)\right) = C. \end{aligned} \tag{7}$$

Let us now rewrite $S_q(j)$, when $q > \alpha$, as:

$$S_q(j) = \frac{1}{n_j}\sum |d_{j,k}|^q = \frac{\sigma_j^q}{n_j^{1-q/\alpha}} \underbrace{\frac{1}{n_j^{q/\alpha}} \sum \left|\frac{d_{j,k}}{\sigma_j}\right|^q}_{A}.$$

The normalised quantity A is asymptotically ($n_j \to \infty$) independent of j. Assuming that the details at fixed scale are i.i.d. (in fact they are stationary and weakly correlated) [2]. we apply the results above with a scale parameter of 1 to find, asymptotically,

$$\mathbb{E}[\log_2 S_q(j)] = \log_2 \sigma_j^q - \log_2 n_j^{1-q/\alpha} + C. \tag{8}$$

From the dyadic sampling of the wavelet coefficient of data n long, we have that $n_j \simeq n2^{-j}$, and from relation (2) ($\sigma_j = \sigma_0 2^{jH}$) we conclude that asymptotically:

$$\mathbb{E}[\log_2 S_q(j)] = \left[1 + q\left(H - \frac{1}{\alpha}\right)\right] j + \text{Cste}. \tag{9}$$

This reveals that for $q > \alpha$ the q^{th} order Logscale Diagram should exhibit an approximately straight line of slope $1 + q\left(H - \frac{1}{\alpha}\right)$. In summary, should an LFSM process be analysed at $q > \alpha$, although the $S_q(j)$ have infinite moments, the $\log_2(S_q(j))$ do not, yielding well defined quantities θ_q (expectations of slopes in Logscale Diagrams), which do not have an interpretation as exponents. The θ_q are equal to $\zeta_q = qH$ over $q \in (-1, \alpha)$, but for $q > \alpha$ they are unrelated to the ζ_q and hence to the multifractal spectrum, and are given by $\theta_q = 1 + q\left(H - \frac{1}{\alpha}\right)$. (Note that a similar statement can be made for $q < -1$, however we do not address this here as it involves difficult issues concerning estimation of negative MF spectral exponents [4].) Finally, note that the Legendre transform of this two-component θ_q is in fact the same as that of ζ_q, except that the left hand branch of the transform terminates with a jump to $-\infty$.

4. DISCUSSION: THE ESTIMATION TRAP

In figure 1 the estimation procedure is successfully applied to the MfBm sample path shown for $q \in [-1, 5]$. In the middle row Logscale Diagrams of orders $q = 1$ and 3 are shown and the estimates are seen to closely follow, to within the confidence intervals, the straight lines representing the theoretical results. In the bottom row the ζ_q estimates also follow the theoretical values closely, and the MF spectrum is well reproduced. In figure 2 the same procedure is applied to the LFSM process. As expected, straight lines are observed in both LD's even though the underlying S_q is not defined in the $q = 3$ case. In the bottom left plot the estimated slopes θ_q clearly display the predicted change in q dependence at $q = \alpha$, and the estimation is good except near the transition point. In contrast to this, the estimate of the Legendre MF spectrum has a well rounded shape which is quite distinct from the theoretical piecewise linear Legendre transform of *both* ζ_q and θ_q, and is reminiscent of the MF spectrum of the MfBm model. This is the first key impact of applying the standard analysis to a multifractal α-stable process: the measured MF spectrum is indistinguishable from other kinds of multifractals, so that the α-stable and self-similar nature of the process is not detected. This despite the fact that the sources of the multifractal variability is radically different in the two cases. Indeed in the particular case of the Lévy stable process the increments are stationary, and the sample path irregularities inherit purely from its α−SSS nature, in clear contrast with MF cascades where it derives from the complex dependency structure inherent in the cascade hierarchy. The absence of an obvious α-stable signature is notable not only in the misleading estimate of the MF spectrum, but also in the LD's, where well behaved straight lines are found, and in the θ_q, as estimation effects round off the sharp change in slope at $q = \alpha$. The second impact is that the essentially simple nature of the MF spectrum in the LFSM case, being piecewise linear and determined entirely by its α-stable and self-similar natures, is not recovered. The source of these errors is *not* the difference in Legendre transform between ζ_q and θ_q, as these are in fact essentially the same as noted above, but rather poor estimation of the θ_q near $q = \alpha$. The Legendre transform is very sensitive to errors in straight lines, and it is mainly the slight curvature in the estimates in the *correct* range of $q < \alpha$ which distorts the sharp point of the true transform to the bell shaped form. The difficulty, in any case, results from the change in nature of the moments at $q = \alpha$. One way to avoid being mislead in this way is to

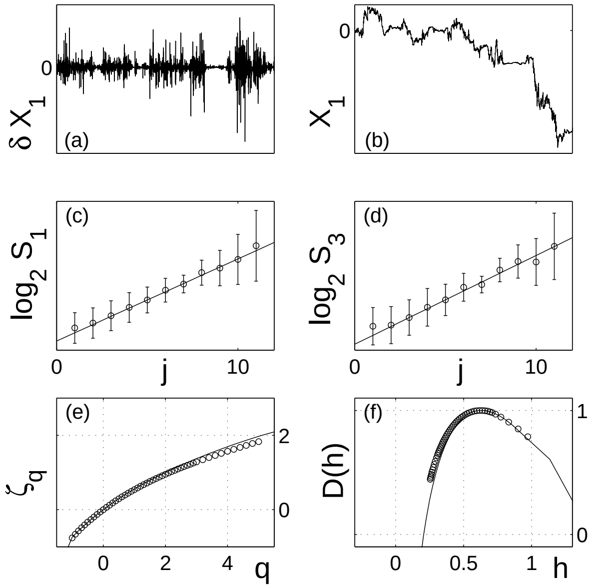

Figure 1: **Fractional Brownian motion in Multifractal time, $H = 1/2$.** (a) Increments, (b) process, (c) LD for $q = 1$ and (d) $q = 3$, (e) partition functions exponents ζ_q, (f) Multifractal spectrum $D(h)$. The dashed line (resp. circles) show for the theoretical (resp. estimated) values. The underlying cascade is binomial with symmetric-β distributed multipliers with $p = 1.6$, [13].

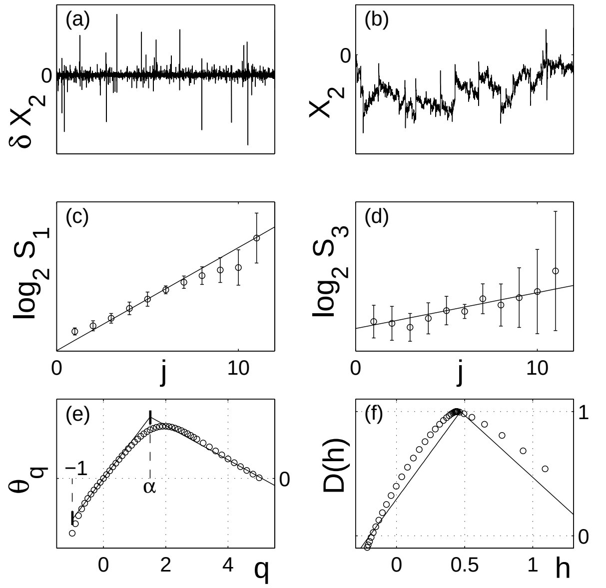

Figure 2: **LFSM process with $\alpha = 3/2$, $H = 0.47$.** As for figure 1 except (e) shows the slopes θ_q, which are only ζ_q if $q \in (-1, \alpha)$. In (f) a rounded MF spectrum estimate is seen, qualitatively different from the true one (lines) .

determine the exact cause of the distorted estimation of the θ_q near $q = \alpha$, an significantly reduce it. Another approach is to prepend an examination of the finiteness of the $S_q(j)$ to the estimation procedure, to detect the nature of the process in advance and determine a range for q which is both relevant for the MF spectrum and safe for estimation. As already noted this cannot be performed in the LD's, where all moments are finite. Such tests will typically be difficult to perform on the series themselves, as they may be non-stationary and/or exhibit strong and long-range statistical dependencies. The wavelet coefficients reproduce the behaviour of moments, yet are stationary within scales and display weak or short range dependencies [1, 2]. They therefore constitute likely candidates on which to base such tests. Each of these approaches is being investigated.

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