Scale Invariant
Infinitely Divisible Cascades

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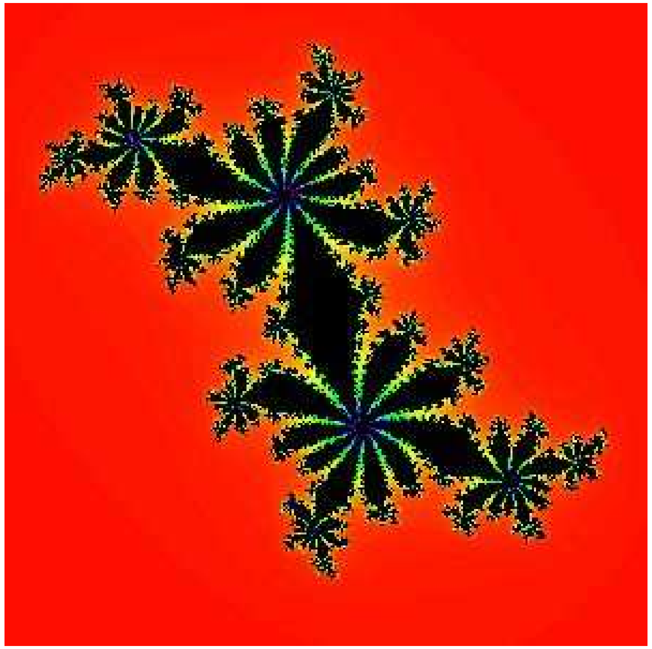
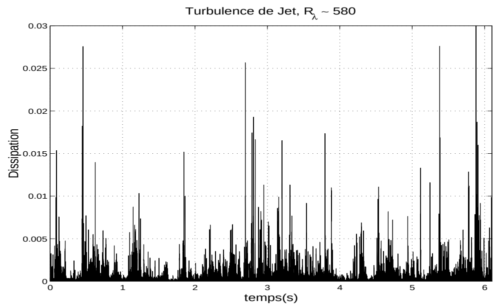
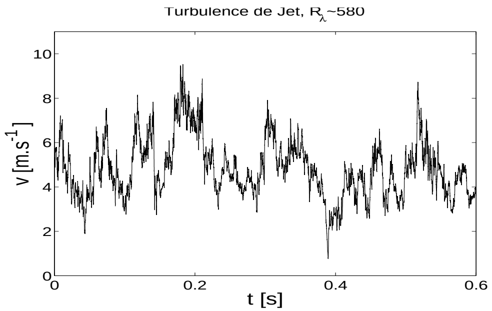
Scale invariance







ANALYSIS / MODELING of **irregular** objects

multifractal scaling $\mathbb{E}\,\lvert\delta_\tau X\rvert^q \sim \tau^{\zeta(q)}$	simulation of irregular objects
multiplicative cascade phenomenology	multiplicative cascade algorithms
(Richardson cascade in turbulence)	(binomial cascades. . .)

Usual scale invariant models

☛ **SELF-SIMILARITY** (stat. incr.)

$$\mathbb{E}\,|\delta_\tau X|^q = \mathbb{E}\,|X(1)|^q \cdot \tau^{qH}$$

fractional Brownian motion,
Linear Fractional Stable Motion...

☛ **MULTIFRACTAL FORMALISM**

$$\mathbb{E}\,|\delta_\tau X|^q = c_q\,\tau^{\zeta(q)}$$

binomial cascades,
cascades on wavelet coefficients,
Multifractal Random Walk (MRW)...

Limiting properties...

– only few $\zeta(q)$ can be obtained,
– scaling for DISCRETE SCALES only,
– NON-STATIONARY increments,
– processes defined for DISCRETE TIME only.

Log-infinitely divisible cascades (LIDC)

$$\mathbb{E}\,|\delta_\tau X|^q = c_q \exp[-H(q)\cdot n(\tau)]$$

✔ $n(\tau) = -\log\tau \implies \mathbb{E}\,|\delta_\tau X|^q = c_q\tau^{H(q)} \equiv c_q\tau^{\zeta(q)}$
(scale invariance)

✔ $n(\tau) \neq -\log\tau \implies$ a hint to non-scale invariant analysis :

– turbulence `Castaing et al. 1990`
– internet traffic `Veitch et al. 2000`

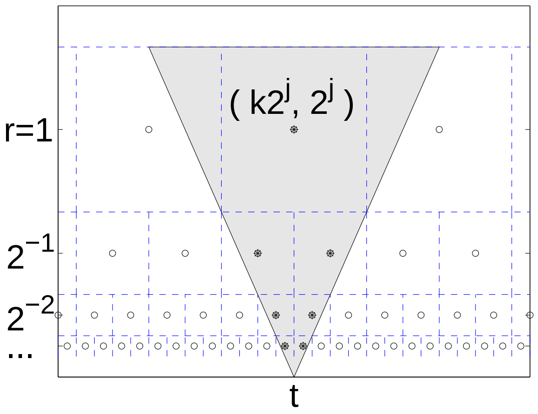
✔ multiplicative interpretation

How to build a signal obeying a LIDC ?

moreover with nice properties such as *stationary increments*, etc. . .

From binomial cascades to Infinitely Divisible Cascades

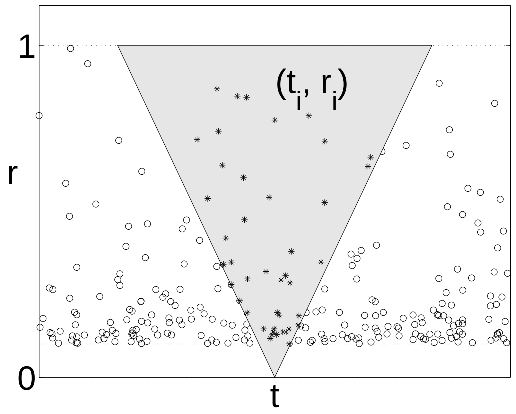
deterministic
distribution of points



$$Q_r(t) = \prod_j \Lambda_j(t),$$

only $\begin{cases} t_{k,j} = k2^j \\ r_j = 2^j \end{cases}$

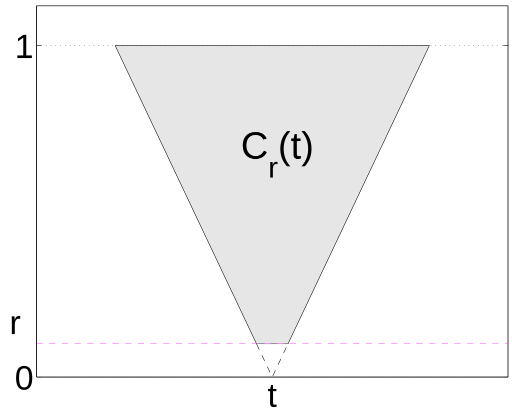
random
distribution of points



$$Q_r(t) \propto \prod_{(t_i,r_i)} W_i$$

$\begin{cases} t_i : \text{uniform} \Rightarrow \text{stationary} \\ r_i : 1/r \Rightarrow \text{scaling} \end{cases}$

infinitely divisible
random measure



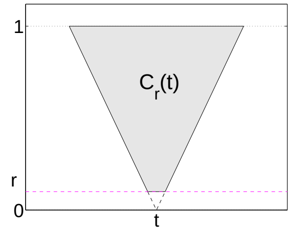
$$Q_r(t) \propto \exp M(\mathcal{C}_r(t))$$

$\equiv$ CONTINUOUS MULT. CASCADE

Infinitely Divisible Cascading noise

- G = infinitely divisible distribution, moment generating function $\tilde{G}(q) = e^{-\rho(q)}$,
- $dm(t,r) = \dfrac{c\,dt\,dr}{r^2}$ positive measure on time-scale plane,
- $M(\mathcal{C}_r(t))$: moment gen. fun. $\exp\left[-\rho(q)\, m(\mathcal{C}_r)\right] = r^{\rho(q)}$

$$Q_r(t) = \frac{\exp\left[M(\mathcal{C}_r(t))\right]}{\mathbb{E}[\exp M(\mathcal{C}_r(t))]}$$



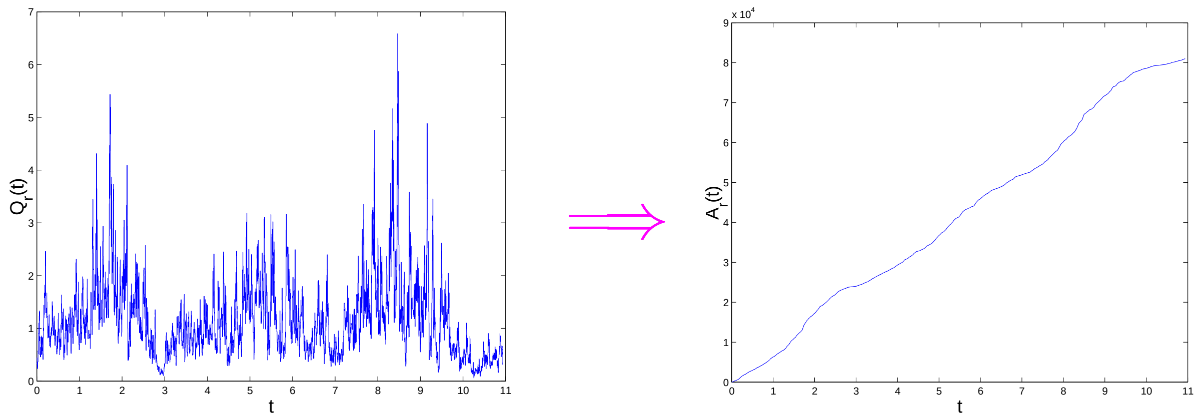
$Q_r(t)$ is stationary

and $\varphi(q) = \rho(q) - q\rho(1) \Longrightarrow \boxed{\mathbb{E}[Q_r^q] = r^{c\varphi(q)}}$

Infinitely Divisible Cascading motion

Pb : Q_r degenerates as $r \to 0 \ldots$

Solution : $A_r(t) = \int_0^t Q_r(s)\, ds \Longrightarrow \mathbb{E}A_r(t) = t$



$$A(t) = \lim_{r \to 0} A_r(t) \quad \ldots$$ has stationary increments

and $\boxed{\mathbb{E}\delta_\tau A^q \sim \tau^{q+c\varphi(q)}}$

Infinitely Divisible Cascading random walk

$A(t)$ an IDC motion B_H a **f**ractional **B**rownian **m**otion,(Hurst parameter H)

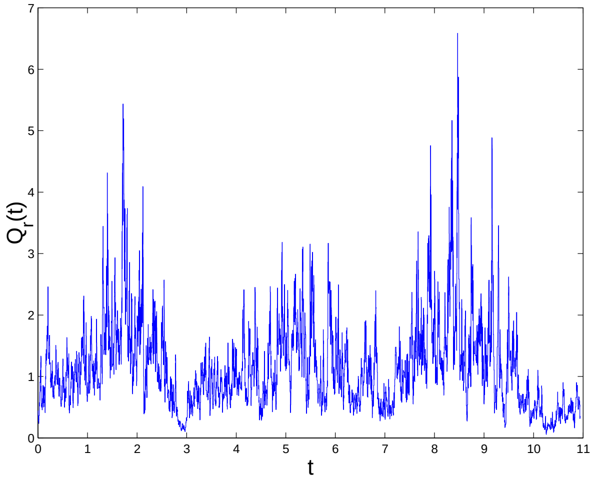
$$\Longrightarrow \quad V_H(t) = B_H(A(t)), \quad t \in \mathbb{R}^+$$

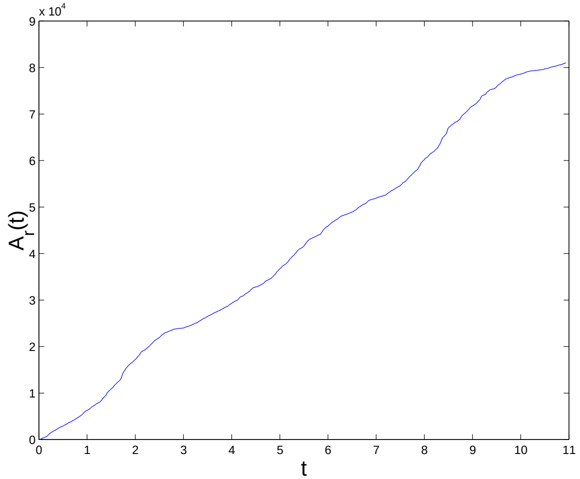
<u>Reminding that</u>
$$\begin{cases} B_H \text{ has stationary increments} \\ \forall\, t \in \mathbb{R}^+, \mathbb{E}|B_H(t)|^q = t^{qH} \cdot \mathbb{E}|B_H(1)|^q, \end{cases}$$

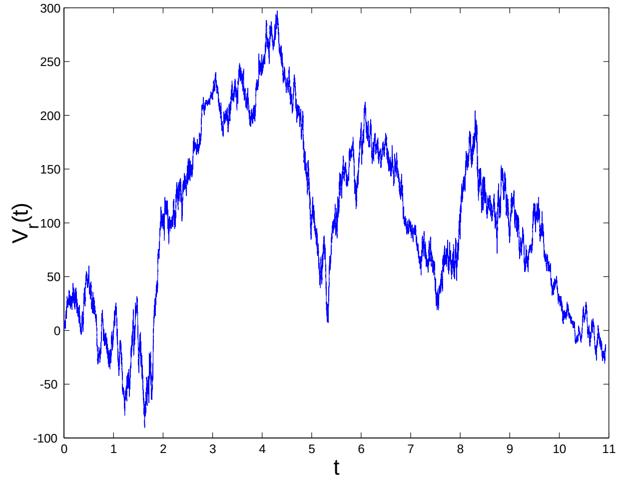
V_H has stationary increments

and $$\boxed{\mathbb{E}|\delta_\tau V_H|^q \sim \tau^{qH + c\varphi(qH)}}$$

Example of an Infinitely Divisible Cascade







$$Q_r(t) = \frac{\exp\left[M(\mathcal{C}_r(t))\right]}{\mathbb{E}[\exp M(\mathcal{C}_r(t))]} \quad A_r(t) = \int_0^t Q_r(s)\, ds$$

$$\downarrow r \to 0$$

$$A(t) \quad \Rightarrow \qquad V_H(t) = B_H(A(t))$$

Example of application : She-Lévêque model of turbulence

She-Lévêque $\Leftrightarrow$ pure Poisson cascade $\in$ Inf. Div. Casc.

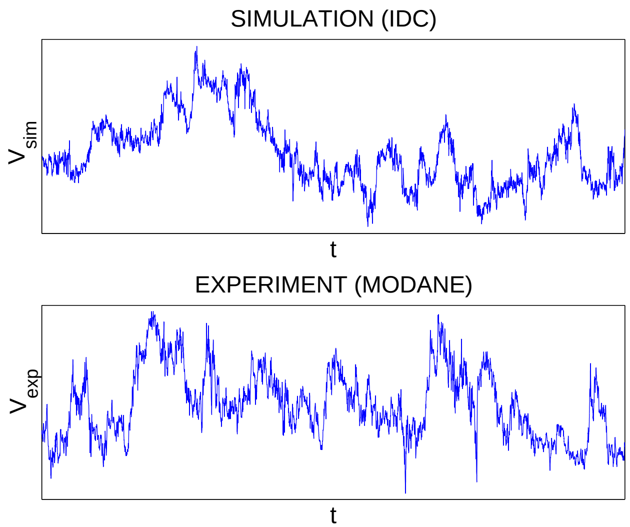
$$\varepsilon_r = \frac{1}{\Omega_r}\int_{|\mathbf{x}|<r} \epsilon(\mathbf{x},t)d\mathbf{x} \quad \Longrightarrow \quad \mathbb{E}\varepsilon_r^q \propto r^{\tau(q)}$$

$$\delta v_r = v(x+r,t) - v(x) \quad \overset{Kolmogorov62}{\Longrightarrow} \quad \mathbb{E}|\delta v_r|^q \propto r^{q/3+\tau(q/3)}$$

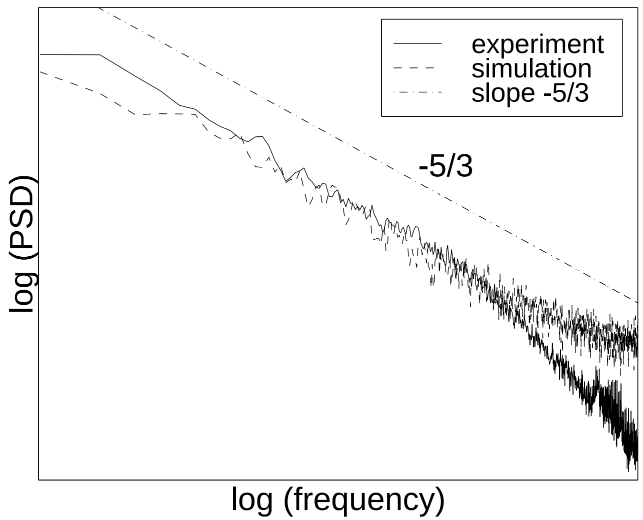
. . . suggests the correspondence $\begin{cases} \varepsilon_r \leftrightarrow Q_r \\ v \leftrightarrow V_H = B_H(A(t)) \end{cases}$

$$\begin{cases} \mathbb{E}\ \varepsilon_r^q \equiv \mathbb{E}\ Q_r^q \sim r^{-\frac{2}{3}q+2\left(1-\left(\frac{2}{3}\right)^q\right)} \\ \mathbb{E}\ |\delta v_r|^q \equiv \mathbb{E}\ |\delta_\tau V_{1/3}|^q \sim \tau^{\frac{1}{9}q+2\left(1-\left(\frac{2}{3}\right)^{q/3}\right)} \end{cases}$$

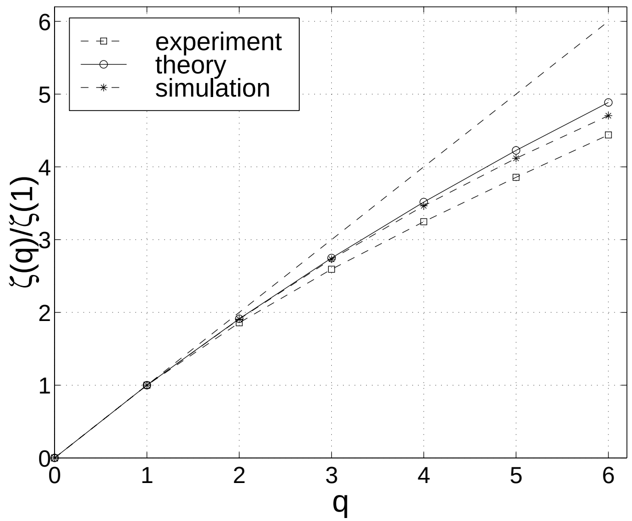
IDC *vs* She-Lévêque model and experiment



IDC signal vs experiment



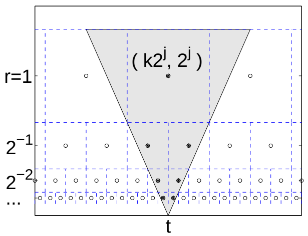
Power density spectra

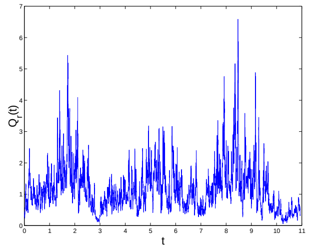


Scaling exponents

$\Longrightarrow$ artificial signal obeying She-Lévêque model !

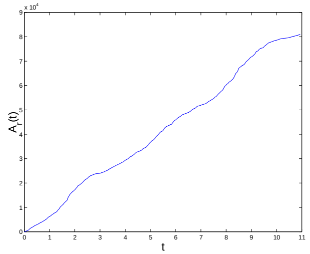
DISCRETE multiplicative cascade





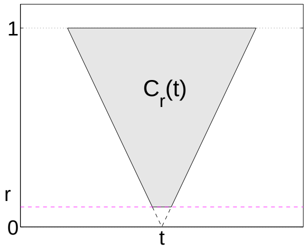
$$\mathbb{E}[Q_r^q] = r^{c\varphi(q)}$$

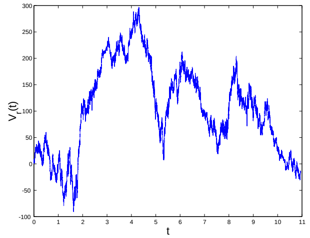
$\Downarrow$



$$\mathbb{E}\delta_\tau A^q \sim \tau^{q+c\varphi(q)}$$

CONTINUOUS multiplicative cascade





$$\mathbb{E}|\delta_\tau V_H|^q \sim \tau^{qH+c\varphi(qH)}$$

- stationary increments
- continuous time $(t \in \mathbb{R}^+)$
- continuous scale invariance

- any Inf. Div. scaling exponents
- MATLAB synthesis procedures

Perspectives

- ✔ **analysis :** test of tools (estimation, prediction...)
- ✔ **turbulence**, **internet traffic**, **finance**, **clouds**...
- ✔ **math :** are there other multifractals than IDCs ?
- ✔ **multidimensional** IDC ?
- ✔ **non scale invariant cascades** ?...

References

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