Advances in Turbulence X
Proceedings of the Tenth European Turbulence Conference
H. I. Andersson & P.-Å. Krogstad (Eds.)

Limitation of Scaling Exponent Estimation in Turbulence

B. Lashermes[1], C. Baudet[2], P. Abry[1], P. Chainais[3]
[1] CNRS, Laboratoire de Physique, École Normale Supérieure, Lyon, France
[2] LEGI, Université Joseph Fourier, Grenoble, France
[3] CNRS, ISIMA-LIMOS, Université Blaise Pascal, Aubière, France
Contact address: *bruno.lashermes@ens-lyon.fr*

• **Estimation of scaling exponents in Turbulence ?** A significant characteristic of fully developed turbulence is *scale invariance*, i.e., in a wide range of scale ratios, usually known as the *inertial range*, the moments of order $q > 0$ of the increments of the velocity field $v(x)$ behave as power laws with respect to scale ratios a (see e.g., [2]):

$$\mathbb{E}|v(x + ax_0) - v(x)|^q \quad = \quad c_q|a|^{\zeta(q)}. \tag{1}$$

A key issue in the analysis of turbulence data lies in accurately and precisely measuring the corresponding scaling exponents. Strange though it may seem, the question of the precise statistical performance of the (standard multiresolution based) estimators for these exponents has been mostly overlooked (cf., a contrario, [11], [7]) and this is precisely the issue carefully addressed here.

First, we benchmark the statistical performance of the $\zeta(q)$ estimators by applying them to a large number of numerical replications of theoretically controlled multifractal processes recently introduced in the literature [1]. Our key result is that multiresolution based estimators for $\zeta(q)$ undergo a generic *linearisation effect*: there exists a critical order q_*^+ value below which the estimators correctly account for the scaling exponents and above which they significantly depart from the $\zeta(q)$ and necessarily behave as a linear function in q. We show that this is not a finite observation duration effect but that it is deeply rooted in the multiplicative nature of the processes. Second, we apply the estimators to empirical turbulence data, we observe similar linearisation effects and estimate the corresponding critical q_*^+ value. We comment on the implied limitations in the estimation of scaling exponents and consequences in turbulence.

• **Estimation procedures.** The standard multiresolution based estimation procedures (MRA) $\hat{\zeta}(q, n)$ consist of three steps. First, one computes multiresolution quantities $T_X(a, t) = \langle \psi_{a,t}, X \rangle$, where $\psi_{a,t}(u) = 1/a\psi((u - t)/a)$ are dilated (with scale factor a) and translated (around time t) templates of a reference pattern ψ and where X is the process to be analysed. Depending on the choice of ψ, the $T_X(a, t)$ amounts to box-aggregated or increment or wavelet coefficients. Second, one computes the qth-order structure functions, defined as time averages of the $|T_X(a, t)|^q$, at scale a: $S_n(a, q) = \frac{1}{n_a} \sum_{k=1}^{n_a} |T_X(a, t_k)|^q$ where n is the process length, and n_a the number of coefficients $T_X(a, t_k)$ available at scale a. When X presents scaling as in Eq. (1), one has : $S_n(a,q) \sim c_q|a|^{\zeta(q)}$. Third, estimates $\hat{\zeta}(q,n)$ are obtained from linear regressions in $\log S_n(a,q)$ vs $\log a$ diagrams. The range of qs is restricted to $q > 0$ to avoid technical discussions fully outside the scope of this paper.

• **Multifractal reference processes.** To benchmark the statistical performance of the $\hat{\zeta}(q,n)$, we apply them to synthetic multifractal processes that can be considered as a satisfactory model for velocity turbulence data: fractional Brownian motion in multifractal time (FBM(MT)) [6]. FBM(MT) $V_H(t)$ is defined combining fractional Brownian motion with Hurst parameter H, $B_H(t)$, and a multiplicative cascade density $Q_r(t)$, as $V_H(t) = B_H(A(t))$ with $A(t) = \lim_{r\to 0}\int_0^t Q_r(s)ds$. Instead of constructing $Q_r(t)$ from the celebrated Mandelbrot's multiplicative cascades [5], we use compound Poisson cascades (CPC), recently defined by Barral & Mandelbrot [1]. CPC present improved statistical properties (*continuous* scale invariance and *full* stationarity) still with prescribed scaling as in Eq. (1) with known $\zeta(q)$ exponents that depend jointly on H and the cascade multiplier characteristics. The Legendre transform $D(h) \equiv 1 + \min_q\ (qh - \zeta(q))$ of $\zeta(q)$ will be used farther.

• **Estimation statistical performance: linearisation effect.** First, we observe that, for each and every replication of the analysed process, there exists a finite range of q values, denoted $[0, q_0]$, within which $\hat{\zeta}(q,n)$ accounts for the theoretical value $\zeta(q)$, while outside that range, i.e., when $q > q_0$, $\hat{\zeta}(q,n)$ significantly departs from $\zeta(q)$ and systematically present a linear behaviour in q (cf. Fig. 1, (top left)). Moreover, these q_os values are spread around a critical q_*^+ defined in (3). Second, to further study this linear in q behaviour of the $\hat{\zeta}(q,n)$, one computes the Legendre transforms $\hat{D}(h,n)$ of the $\hat{\zeta}(q,n)$. Fig. 1 (top right) shows that each $\hat{D}(h,n)$ is abruptly ended by an accumulation point, (h_0, D_0), and hence accounts for the theoretical $D(h)$ only when $h \geq h_0, \hat{D} > D_o$. Furthermore, the accumulation points are spread around a critical point (h_*^+, D_*^+), defined as the (left) zero of $D(h)$ (cf. Eq. (2)). Third, numerical simulations not reported here [3] show that the critical values q_*^+, h_*^+, D_*^+ depend neither on the resolution or depth of the underlying multiplicative cascade nor on the observation duration length n or on the number of integral scales, in other words, this is not a finite size effect. They also show that this generic and systematic effect is observed for *all known synthetic multifractal processes* and with *all MRA based estimators*. It will be referred to as *linearisation effect*.
These empirical results lead us to formulate the following conjecture. MRA based $\hat{\zeta}(q,n)$ behave as:

$$\begin{cases} q \in [0, q_0], & \hat{\zeta}(q,n) \to \zeta(q) \\ q \geq q_0, & \hat{\zeta}(q,n) = 1 - D_o + h_o q \to 1 - D_*^+ + h_*^+ q, \end{cases} \tag{2}$$

where the average values q_*^+, h_*^+, D_*^+ of q_o, h_o, D_o are given by:

$$q_*^+ \ /\ q\zeta'(q) - \zeta(q) \geq -1 \text{ if } q \in [0, q_*^+]\ ;\ h_*^+ \ /\ D(h_*^+) = 0 = D_*^+. \tag{3}$$

It is worth noting that this critical q_*^+ and hence the linearisation effect is not related to any statistical moment divergence issue as sometimes written in the literature. This criterion had been obtained previously in the literature [7], [8], but only in the case of Mandelbrot cascades and box-aggregated estimators that do not apply to velocity signals. We extend it to all known multiplicative cascade schemes and multifractal processes and all MRA based $\hat{\zeta}(q,n)$ [3], showing that the linearisation effect is a very generic and systematic effect in the scaling exponent estimation of multiplicative processes. A practical procedure for the estimation of q_*^+, hereafter denoted $\hat{q}_*^+$, has been developed, and numerically probed on synthetic prossesses, see [4].

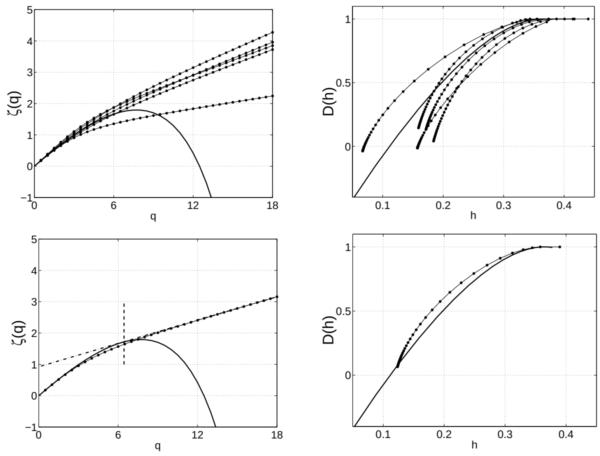

Figure 1: **Linearisation effect on synthetic data.** Data consist in FBM(MT) built on CPC cascades, with 2^2 integral scales and $n = 2^{15}$. $\zeta(q)$ (solid lines) and $\hat{\zeta}(q,n)$ (dotted lines) (left column) and corresponding Legendre transforms (right column), for 5 replications (top raw) and averaged over 100 replications (bottom raw). One can clearly observe the linearisation effect and the corresponding accumulation point. The vertical dashed line (bottom left) denotes the theoretical q_*^+.

• **Application to turbulence.** We apply the $\hat{\zeta}(q,n)$ and $\hat{q}_*^+$ procedures to two sets of hot-wire velocity data: jet turbulence with $R_\lambda \sim 580$ [10], wind tunnel cryogenic gaseous helium (T = 4K) turbulence with $R_\lambda \sim 3200$ [9]. Data consist in 69 runs (resp., 25) with 60 (resp., 60) integral scales and $n = 2^{20}$ (resp., $n = 2^{22}$). Linear fits are performed in the usual inertial range. We observe, first, that a linearisation effect occurs on empirical data (cf. Fig. 2), that is highly similar to that obtained on synthetic FBM(MT). Second, $\hat{q}_*^+$ yields 9.4 ± 0.4 ($R_\lambda \sim 580$) and 9.2 ± 0.5 ($R_\lambda \sim 3200$). Third, the $\hat{\zeta}(q,n)$ for $q < \hat{q}_*^+$, are satisfactorily modelled either with the log-normal model — with the commonly accepted value for the intermittency parameter $C_2 \simeq 0.025$ —, or the log-Poisson She-Lévêque model (no free parameter). For these two celebrated models [2], the theoretical q_*^+ can be computed from Eq. (3), yielding respectively 8.94 and 12.36, respectively, in reasonable agreement with our empirical findings.

• **Conclusions.** For velocity turbulent data, it is not possible, pointless and meaningless to estimate scaling exponents $\zeta(q)$ beyond a critical order $q_*^+ \simeq 9.5$.

This q_*^+ does not depend on the Reynolds number R_λ, which is consistent with the fact that the function $\zeta(q)$ is expected to be universal (i.e., independent of R_λ). This linearisation effect and the corresponding q_*^+ do not result from estimation difficulties, this will not disappear with *better* experiments (higher resolution hot wire probes, longer observation duration, ...): it is deeply rooted in the nature of the data and is therefore intrinsic to the phenomenon under study. Ongoing works further extend these results to dissipation and other experimental and numerical data sets.

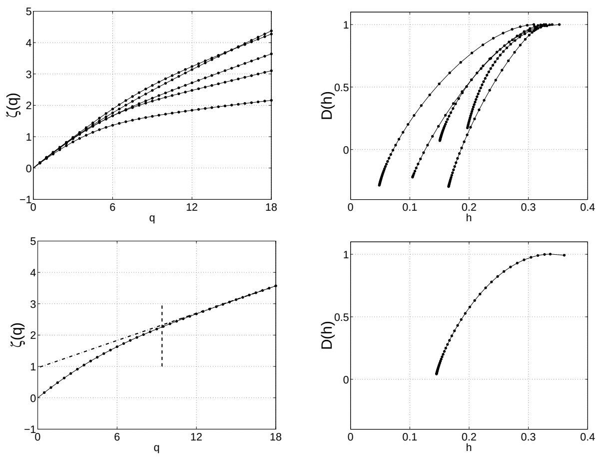

Figure 2: **Linearisation effect on velocity turbulence data.** $\hat{\zeta}(q,n)$ (left column) and corresponding Legendre transforms (right column) for 5 runs (top raw) and averaged over 69 runs (bottom raw), with $R_\lambda \sim 580$. Linearisation effect and corresponding accumulation points are similar to that observed on synthetic data (Vertical dashed line, bottom left, denotes $\hat{q}_*^+$). Similar plots and effects are obtained with $R_\lambda \sim 3200$.

References

[1] J. Barral and B. Mandelbrot, *Probab. Theory Relat. Fields*, 124:409–430, 2002.

[2] U. Frisch, *Turbulence, the Legacy of A.N. Kolmogorov.* Camb. Univ. Press, 1995.

[3] B. Lashermes, P. Abry, and P. Chainais, "New insights on the estimation of scaling exponents," 2003. preprint.

[4] B. Lashermes, P. Abry, and P. Chainais, Proc. of ICASSP, Canada, 2004.

[5] B. B. Mandelbrot, *J. Fluid. Mech.*, 62:331, 1974.

[6] B. B. Mandelbrot, *Scientific American*, 280:70–73, Feb. 1999.

[7] G. M. Molchan, *Phys. Fluids*, 9(8): 2387–2396, 1997.

[8] M. Ossiander and E. Waymire, *The Annals of Statistics*, 28(6)1533–1560, 2000.

[9] S. Pietropinto *et al.*, *Physica* C, 386:512–516, 2003.

[10] G. Ruiz Chavarria, C. Baudet, and S. Ciliberto, *Phys. Rev. Lett.*, 74:1986–1989, 1995.

[11] F. Schmitt, D. Schertzer, S. Lovejoy, and Y. Brunet, *Nonlinear Processes in Geophysics*, 1:95–104, 1994.