SCALE INVARIANT INFINITELY DIVISIBLE CASCADES

P. Chainais (1), *R. Riedi* (2), *P. Abry* (3)

(1) ISIMA-LIMOS UMR 6158 - Université Blaise Pascal, Aubière, France
pchainai@isima.fr, www.isima.fr/∼chainais
(2) Dept of ECE, Rice University, Houston Texas, USA
riedi@rice.edu, www.rice.edu/∼riedi,
(3) CNRS UMR 5672, Laboratoire de Physique. ENS Lyon, France,
Patrice.Abry@ens-lyon.fr, www.ens-lyon.fr/∼pabry.

ABSTRACT

Multiplicative processes and multifractals proved useful in various applications ranging from hydrodynamic turbulence to computer network traffic. It was recently shown and explained how and why multifractal analysis could be fruitfully placed in the general framework of infinitely divisible cascades. The aim of this contribution is to design processes, called Infinitely Divisible Cascading (IDC) noise, motion, and random walk. These processes possess at the same time stationary increments as well as multifractal and more general infinitely divisible scaling that can be prescribed a priori over a continuous range of scales. This communication focuses on the specific scale invariant case. To illustrate the powerfulness of the method, we mention that IDC processes can exactly mimic the scaling behaviors predicted by the celebrated She-Lévêque model of turbulence. MATLAB routines implementing those processes are available from our Web pages.

1. MOTIVATION

Scale invariance and related phenomena have received considerable attention in the past from the point of view of both analysis and modelling. Various kinds of scaling form an indisputable component of empirical data observed in a wide variety of applications ranging from natural phenomena (hydrodynamic turbulence [1], biology and body rhythms [2]. . .) to purely human phenomena created by mankind's activities (computer networks [3, 4], financial markets [5, 6]. . .). Often, the presence of scaling in the data can be tied to crucial properties of the system, e.g., high volatility in markets and large waiting queues in computer networks.

Most prominently, self-similar processes have been favored as models for scale invariance for their simplicity. Indeed, any self-similar process $X(t)$ with stationary increments spots an appealing and simple scale invariance. Let $\delta_\tau X(t) = X(t+\tau) - X(t)$ denote its increments over a lag τ, then

$$\mathbb{E}|\delta_\tau X(t)|^q = c_q \cdot |\tau|^{qH}, \qquad (1)$$

where H is the Hurst parameter. To provide processes with more realistic scaling which are able to match real world data, multiplicative cascades and the framework of multifractal analysis were introduced, allowing a non-linear dependence on the order q of exponents $\zeta(q) \neq qH$ in (1) so that:

$$\mathbb{E}|\delta_\tau X(t)|^q = c_q \cdot |\tau|^{\zeta(q)}. \qquad (2)$$

The synthesis of processes that possess a priori prescribed multiscaling properties as well as other casual characteristics such as second-order stationarity of the increments or a continuous rather than discrete scaling region proved extremely difficult. Partial solutions have been provided by several authors [7, 8, 9, 10]. This is a contribution to the pavement of this difficult path [11]. This paper sets off to construct new processes called Infinitely Divisible Cascading noise, motion, and random walk with flexible and natural scaling properties. The progression of moments may depend in a non linear way on the order q, and can be determined between arbitrary scales. Moreover, it may depend in an arbitrary way on scale, not necessarily in form of a power law. To this purpose, one can place multifractal analysis in the more general framework of infinitely divisible cascades (IDC) [12, 13] characterized by:

$$\mathbb{E}|\delta_\tau X(t)|^q = c_q \exp[-H(q) \cdot n(\tau)]. \qquad (3)$$

Note how the framework of LIDC encompasses scaling in the form of power laws by setting $n(\tau) = -\log(\tau)$. The extra degree of freedom in scale dependence was found highly useful for the analysis and modelling of empirical data in turbulence [14] and computer network traffic [15]. For sake of simplicity, this paper focuses on the scale invariant case only.

2. INFINITELY DIVISIBLE CASCADING NOISE

We introduce through the following notion a generalization of iterative multiplication. The following Infinitely Divisible Cascading noise can be seen as a generalization of the "product of pulses" by Barral & Mandelbrot [8] using ideas due to Schmitt & Marsan [10].

Definition 1. *Let G be an infinitely divisible distribution with moment generating function $\tilde{G}(q) = e^{-\rho(q)}$. Let $dm(t,r) = g(r)dtdr$ a positive measure on the time-scale half-plane $\mathcal{P}^+ := \mathbb{R} \times \mathbb{R}^+$. This choice of a time-invariant* control measure m

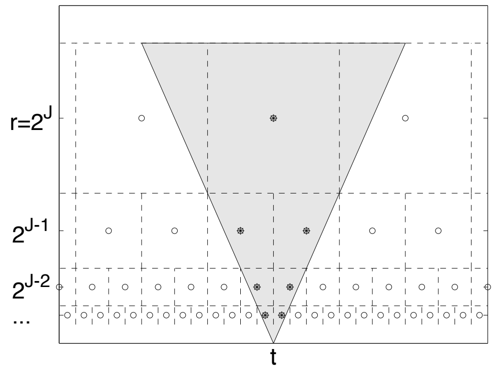

Fig. 1. **Binomial multiplicative cascade.** The construction of the Binomial cascade follows a rigid dyadic, deterministic geometry; the density is an iterative product of pulses located in the time-scale half-plane at $(k/2^n, 2^n)$.

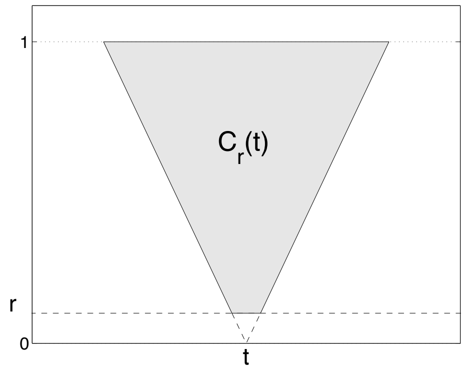

Fig. 2. **Infinitely Divisible Cascading.** For the Infinitely Divisible Cascade the geometry is random, stationary in time and continuous in scale in the time-scale half-plane.

will ensure stationarity. To provide scale dependence, a so-called cone of influence $\mathcal{C}_r(t)$ *is defined for every* $t \in \mathbb{R}$*, such that* $\mathcal{C}_r(t) = \{(t', r') : r \leq r' \leq 1, t - r'/2 \leq t' \leq t + r'/2\}$ *(see Figure 2).*
An Infinitely Divisible Cascading noise *(IDC-noise) is a family of processes* $Q_r(t)$ *parametrized by* r *of the form*

$$Q_r(t) = \frac{\exp\left[M(\mathcal{C}_r(t))\right]}{\mathbb{E}[\exp M(\mathcal{C}_r(t))]}. \quad (4)$$

Here, the time-scale half-plane $\mathcal{P}^+$ *is endowed with the infinitely divisible, independently scattered random measure* M *distributed by* G *and associated to its so-called control measure* $dm(t, r)$*. This measure* M *imprints scaling structure as well as marginal distributions on the cascade.*

The IDC-noise can be recognized as a "continuously iterative" multiplication (compare Figure 1 and Figure 2). One major property of infinitely divisible cascading noises is:

$$\mathbb{E}[Q_r^q] = \exp\left[-\varphi(q)\, m(\mathcal{C}_r)\right] \quad (5)$$

where $\varphi(q) = \rho(q) - q\rho(1)$ and $m(\mathcal{C}_r) = \iint_{\mathcal{C}_r} g(r)drdt$. Note the similarity between (5) and (3).

3. INFINITELY DIVISIBLE CASCADING MOTION AND RANDOM WALK

Where there is noise, there must be motion. Analogous to the theory of T-Martingales and of multiplicative cascades in general we will be interested in the distributional convergence of Q_r. To this end we define:
Definition 2. *An* Infinitely Divisible Cascading motion *(IDC-motion)* $A(t)$ *is the limiting integral of an IDC-noise* $Q_r(t)$*:*

$$A(t) = \lim_{r \to 0} \int_0^t Q_r(s)ds. \quad (6)$$

We establish sufficient conditions for convergence of an IDC in $\mathcal{L}_2$ in [11].

Noting that the process A is non-decreasing, one can obtain a random walk with multifractal properties; following an idea which goes back to Mandelbrot [6] we define an infinitely divisible random walk V_H as follows:
Definition 3. *Let* A *be an infinitely divisible cascading motion, and* B_H *the fractional Brownian motion with Hurst parameter* H*. The process*

$$V_H(t) = B_H(A(t)), \quad t \in \mathbb{R}^+, \quad (7)$$

is called an Infinitely Divisible Cascading random walk *(IDC random walk).*

4. SCALE INVARIANT IDC

Of special interest is the scale invariant case. Exact power law behavior for the moments of Q_r is recovered for the choice of the scale invariant control measure $dm(t, r) = cdrdt/r^2$:

$$\mathbb{E}[Q_r^q(t)] = r^{c\varphi(q)} \quad (8)$$

We study the scaling of moments of scale invariant IDC and establish (see [11]):
Theorem. Let $A(t)$ be an IDC-motion with control measure $dm(t, r) = cdrdt/r^2$.

Then, under some technical assumptions, there exist constants $\overline{C}_q$, $\underline{C}_q$ and $\overline{C}'_q$, $\underline{C}'_q$ such that

$$\underline{C}_q t^{q+c\varphi(q)} \leq \mathbb{E}A(t)^q \leq \overline{C}_q t^{q+c\varphi(q)}. \quad (9)$$

$$\underline{C}'_q t^{qH+c\varphi(qH)} \leq \mathbb{E}|V_H(t)|^q \leq \overline{C}'_q t^{qH+c\varphi(qH)}. \quad (10)$$

We point out that $A(t)$ and $V_H(t)$ exhibit very *rich scaling properties* that can be observed *over a continuous range of scales*. Moreover, they possess *stationary increments* so that one get immediately from (9) and (10):

$$\begin{array}{ccc} \mathbb{E}\delta_\tau A^q & \sim & \tau^{q+c\varphi(q)} \\ \mathbb{E}|\delta_\tau V_H|^q & \sim & \tau^{qH+c\varphi(qH)} \end{array} \quad (11)$$

where $\delta_\tau X(t) = X(t + \tau) - X(t)$.

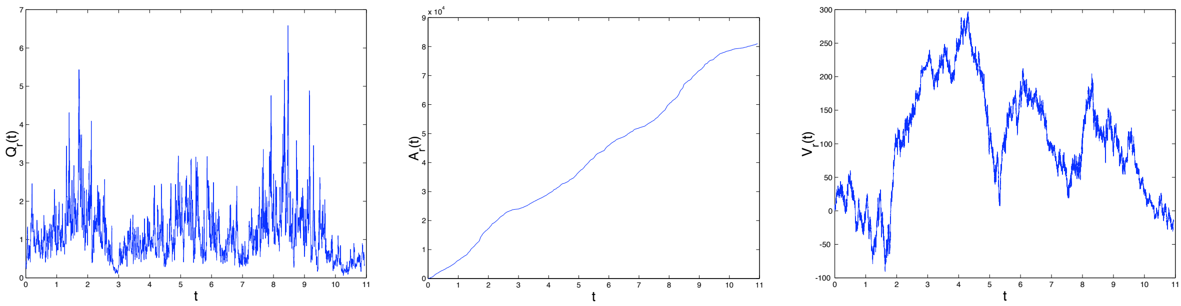

Fig. 3. Sample of a realization of $Q_r(t)$ (left), $A(t)$ (middle) and $V_H(t)$ (right).

5. APPLICATION: THE LOG-POISSON MODEL OF TURBULENCE

Noting that the Poisson law is infinitely divisible, the scale invariant Poisson IDC is of particular interest. This section shows in what sense it corresponds to the *scale invariant log Poisson She-Lévêque* model in the field of turbulence [16, 17, 18]. This model received considerable attention in the physics literature as a statistical description of turbulent flows.

More than half a century has been devoted to the quest for a clear understanding of the statistics of fully developed turbulence. Intermittency in fully developed turbulence is commonly characterised by departures from the results of Kolomogorov's 1941 theory [19]. The fundamental ideas underlying this theory are rooted in the phenomenology of the Richardson's cascade [1]. In Richardson's cascade, energy is introduced in the flow at a rate ε (per unit mass) at the top of a hierarchy of eddies of decreasing size. The energy is 'cascading' down to smaller and smaller eddies at the same rate ε, and is eventually entirely dissipated at the bottom of the hierarchy, still at the rate ε. Such a vision refers to self-similarity. Indeed, Kolomogorov [19] predicted self-similar scaling behaviors for the velocity v of the fluid of the form $\mathbb{E}|\delta v_r|^q \sim r^{q/3}$, where $\delta v_r = v(x+r) - v(x)$ is a (longitudinal) velocity difference across a distance r. A large amount of experimental results later showed that the velocity increments rather behave as $\mathbb{E}|\delta v_r|^q \sim r^{\zeta(q)}$ with $\zeta(q) \neq q/3$, which is usually referred to as the intermittency phenomenon or anomalous scaling property. In 1962, Kolmogorov [20] introduced the *refined similarity hypothesis* that relates this non linear behavior of exponents $\zeta(q)$ to fluctuations of the locally averaged dissipation $\varepsilon_r = \int_{|\mathbf{x}|<r} \varepsilon(\mathbf{x}, t) d\mathbf{x}$. If scaling of the form $\mathbb{E}\varepsilon_r^q \propto r^{\tau(q)}$ are assumed, the *refined similarity hypothesis* yields behaviors of the moments of velocity increments of the form $\mathbb{E}|\delta v_r|^q \propto r^{q/3+\tau(q/3)}$.

Back to IDC, the scaling behaviors of both processes A and V_H are completely controlled by the fluctuations of the underlying IDC noise Q_r. In parallel with the refined similarity hypothesis, it becomes tempting to relate Q_r to the locally averaged dissipation ε_r of turbulence. Thus V_H appears as the natural analog to the fluid velocity v. We now compare an artificial IDC random walk V_H built on the She-Lévêque model of turbulence to an experimental velocity signal v in a turbulent flow.

In summary, the She & Lévêque model can be seen as a cascade that takes place through the existence of dissipative structures distributed along scales according to a pure Poisson law and with constant energy transfer ratio w (see [13] for more recent interpretations of this model). This leads to a pure Poisson cascading process for the viscous dissipation ε_r. A corresponding IDC noise Q_r can be built as follows. In this case, the continuous random measure M reduces to a point measure $M(\mathcal{C}_r(t)) = \#(\mathcal{C}_r(t)) \cdot \ln w$: the number of points $\#(\mathcal{C}_r(t))$ falling into the region $\mathcal{C}_r(t)$ is a random Poisson variable with mean $m(\mathcal{C}_r(t))$ and each point has the same weight $\ln w$. From (4), it results that the cascade is at each instant t the product of a random number $\#(\mathcal{C}_r(t))$ of multipliers that are all equal to w. The resulting pure Poisson cascading noise reads as

$$Q_r(t) = \exp\left[(1-w)m(\mathcal{C}_r(t))\right] \cdot w^{\#(\mathcal{C}_r(t))}, \qquad (12)$$

The choice $dm(t,r) = cdrdt/r^2$ ensures that the scaling is in terms of power laws (see (8)). Clearly, the process Q_r spots log-Poisson marginal distributions. We emphasize the fact that the whole randomness of this cascade stems from the random Poisson number of multipliers that are all equal to w.

The IDC motion A as well as the IDC random walk $V_{1/3}$ can be defined from Q_r using the fractional Brownian motion with Hurst parameter $H = 1/3$:

$$V_{1/3}(t) = B_{1/3}(A(t)). \qquad (13)$$

The scaling properties of processes Q_r, A and $V_{1/3}$ are controlled by the choice of w which enters via $\varphi(q) = 1 - w^q - q(1-w)$ into the formerly obtained scaling laws for $r, \tau \leq 1$ (see (5) & (11)):

$$\begin{cases} \mathbb{E}\ Q_r^q & = \quad r^{c[-q(1-w)+1-w^q]} \\ \mathbb{E}\ \delta_\tau A^q & \sim \quad \tau^{(1-c+cw)q+c[1-w^q]} \\ \mathbb{E}\ |\delta_\tau V_{1/3}|^q & \sim \quad \tau^{(1-c+cw)q/3+c[1-w^{q/3}]} \end{cases} \qquad (14)$$

The She-Lévêque model is exactly recovered for $c = 2$ and $w = 2/3$. For $r, \tau \leq 1$:

$$\begin{cases} \mathbb{E}\varepsilon_r^q & \equiv \ \mathbb{E}Q_r^q & \sim \ r^{-\frac{2}{3}q+2\left(1-\left(\frac{2}{3}\right)^q\right)}, \\ \mathbb{E}|\delta v_r|^q & \equiv \ \mathbb{E}|\delta_\tau V_{1/3}|^q & \sim \ \tau^{\frac{1}{9}q+2\left(1-\left(\frac{2}{3}\right)^{q/3}\right)}. \end{cases} \qquad (15)$$

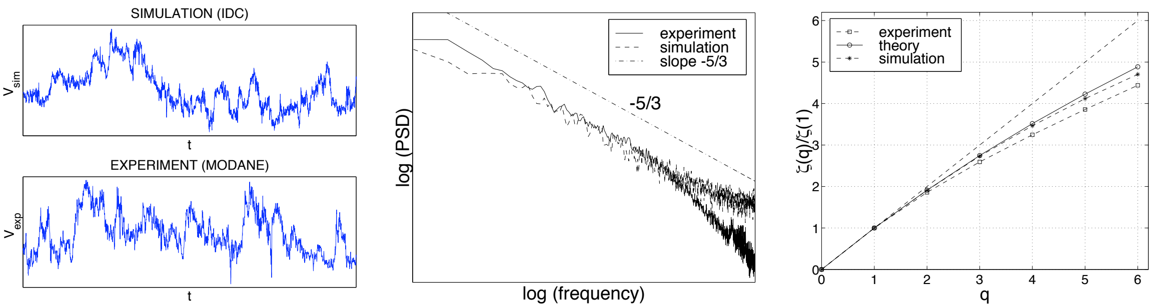

Fig. 4. Comparison between an IDC random walk following She-Lévêque model and an experimental signal (Modane, $Re \sim 2500$): samples of velocity signals (left), power density spectra (middle), relative scaling exponents (right).

Figure 4 shows comparative results obtained from this IDC and from experiment (Modane, courtesy of Y. Gagne [21]). Signals look very similar. Moreover, the IDC random walk $V_{1/3}$ shares many properties of the observed signal. We emphasize that $V_{1/3}$ has stationary increments and exhibits non trivial scaling laws obtained over a continuous range of scales.

To ensure the convergence of process $A(t)$ in $\mathcal{L}_2$, the necessary condition $\mathbb{E}A^2(t) < \infty$ yields (see [11]):

$$1 - \frac{1}{\sqrt{c}} < w < 1 + \frac{1}{\sqrt{c}} \tag{16}$$

If this condition is not obeyed, the cascade either collapses to zero, or explodes to infinity in the $\mathcal{L}_2$ sense. For instance, when $c = 2$ and $w = 2/3 \in [1 - 1/\sqrt{2}, 1 + 1/\sqrt{2}]$, our Poisson cascading noise provides a mathematical model for the physical She-Lévêque model of turbulence.

MATLAB routines are available from our WEB pages.

6. REFERENCES

[1] U. Frisch, *Turbulence. The legacy of A. Kolmogorov*, Cambridge University Press, Cambridge, UK, 1995.

[2] M. Teich, S. Lowen, B. Jost, and K. Vibe-Rheymer, "Heart rate variability: Measures end models," *Preprint*, 2000.

[3] W. Leland, M. Taqqu, W. Willinger, and D. Wilson, "On the self-similar nature of ethernet traffic (extended version)," *IEEE/ACM Trans. Networking*, pp. 1–15, 1994.

[4] K. Park and W. Willinger, Eds., *Self-similar network traffic and performance evaluation*, Wiley, 2000.

[5] B. B. Mandelbrot, *Fractals and Scaling in Finance*, Springer New York, 1997.

[6] B. B. Mandelbrot, "A multifractal walk down wall street," *Scientific American*, vol. 280, no. 2, pp. 70–73, Feb. 1999.

[7] E. Bacry, J. Delour, and J.F. Muzy, "Multifractal random walk," *Phys. Rev. E*, vol. 64, pp. 026103, 2001.

[8] J. Barral and B. Mandelbrot, "Multiplicative products of cylindrical pulses," discussion paper 1287, Cowles Foundation, 1999, submitted to *Probability Theory and Related Fields*.

[9] P. Mannersalo, R. Riedi, and I. Norros, "Multifractal products of stochastic processes: construction and some basic properties," *Applied Probability*, submitted August 2001.

[10] F. Schmitt and D. Marsan, "Stochastic equations generating continuous multiplicative cascades," *Eur. Phys. J. B*, vol. 20, pp. 3–6, 2001.

[11] P. Chainais, R.H. Riedi, and P. Abry, "Infinitely Divisible Cascades," *Submitted*, 2002.

[12] B. Castaing, Y. Gagne, and E. Hopfinger, "Velocity probability density functions of high Reynolds number turbulence," *Physica D*, vol. 46, pp. 177–200, 1990.

[13] P. Chainais, *Cascades log-infiniment divisibles et analyse multirésolution. Application à l'étude des intermittences en turbulence.*, Ph.D. thesis, E.N.S. Lyon, 2001.

[14] P. Chainais, P. Abry, and J.F. Pinton, "Intermittency and coherent structures in a turbulent flow: a wavelet analysis of joint pressure and velocity measurements," *Phys. Fluids*, vol. 11, no. 11, pp. 3524–3539, 1999.

[15] D. Veitch, P. Abry, P. Flandrin, and P. Chainais, "Infinitely divisible cascade analysis of network traffic data," *ICASSP 2000.*

[16] Z. S. She and E. Levêque, "Universal scaling laws in fully developed turbulence," *Phys. Rev.Lett.*, vol.72, no.3, pp.336–339, 1994.

[17] B. Dubrulle, "Intermittency in fully developed turbulence: log-Poisson statistics and scale invariance," *Phys. Rev. Lett.*, vol. 73, pp. 969, 1994.

[18] Z. S. She and E. C. Waymire, "Quantized energy cascade and log-Poisson statistics in fully developed turbulence," *Phys. Rev. Lett.*, vol. 74, no. 2, pp. 262–265, 1995.

[19] A. N. Kolmogorov, in *Turbulence, Classic papers on statistical theory*, S.K. Friedlander and L. Topper, Eds. 1941, pp. 151–161, Interscience publishers.

[20] A. N. Kolmogorov, "A refinement hypothesis concerning the local structure of turbulence in a viscous incompressible fluid at high reynolds number," *J. F. M.*, vol.13, pp.82–85, 1962.

[21] Y. Malécot, *Intermittence en turbulence 3D : statistiques de la vitesse et de la vorticité*, Thèse de doctorat, Université Joseph Fourier - Grenoble I, 1998.