

MULTI-DIMENSIONAL INFINITELY DIVISIBLE CASCADES TO MODEL THE STATISTICS OF NATURAL IMAGES

Pierre Chainais

University of Clermont-Ferrand II
LIMOS - UMR 6158
63173 Aubière Cedex, France

ABSTRACT

Infinitely divisible cascades (IDC) were first introduced in one dimension to provide multifractal time series to model the so-called *intermittency phenomenon* in hydrodynamical turbulence. This work extends the definition of infinitely divisible cascades from 1 to N dimensions, and focuses on the 2D case in particular. 2D IDC appear as good candidates to model the statistics of natural images since they share most of their usual properties. The potential of IDC for applications to image processing (texture synthesis, denoising...) is emphasized.

1. INTRODUCTION

Statistical inference may be of great use for analyzing and understanding images. To this aim, there is a need for probabilistic models of natural images, e.g., to develop Bayesian procedures for object tracking, recognition and analytical performance analysis. Another motivation for pursuing image statistics has been to understand the architecture of animal visual systems. Efficient systems take advantage of statistical structure in their input signals aiming at both denoising and compact representation. In this context [1], an image is treated as a realization of a spatial stochastic process defined on some domain in \mathbb{R}^2 . Part of the difficulty encountered in the search for such models comes from the fact that they should be *non Gaussian* as well as *scale invariant* [1,2].

Previous works –see [1] and references therein– have shown that the power spectrum of an image follows a power law of the form $S(k) \propto 1/k^{2-\eta}$ although η can display some fluctuations. Scaling behaviors have also been observed in a large number of fields including natural phenomena (e.g., turbulence in hydrodynamics) as well as mankind activities (e.g., Internet traffic). As far as higher order statistics are concerned, the multifractal formalism [3] has become one of the most popular framework to analyze signals that exhibit scale invariance. The terms *scale invariance* of a process $X(\mathbf{x})$ then refer to the power law behavior of the moments of some scale dependent quantity¹ built on X . For a positive scalar process $X(\mathbf{x})$ defined on \mathbb{R}^N , one often uses the box averages over a ball of radius r and volume V_r .

$$\varepsilon_r(\mathbf{x}) = \frac{1}{V_r} \int_{\|\mathbf{x}'-\mathbf{x}\|<r} d\mathbf{x}' X(\mathbf{x}'). \quad (1)$$

In short, scale invariance is then described by a set of multifractal

¹For instance increments $X(\mathbf{y}) - X(\mathbf{x})$ in function of $\|\mathbf{y} - \mathbf{x}\|$ or wavelet coefficients $T_X(\mathbf{x}, a)$ at scale a ...

exponents $\tau(q)$ defined through:

$$\mathbf{E}\varepsilon_r(\mathbf{x})^q \propto r^{\tau(q)}, \quad (2)$$

where \mathbf{E} denotes mathematical expectation.

The huge amount of analyses performed in the past 40 years on data from turbulence in fluid dynamics revealed the so called *intermittency phenomenon* [4]: scaling exponents $\tau(q)$ of the local energy dissipation exhibit a non linear dependence on q . This observation was one of the main motivations which gave birth to the multifractal formalism. Note that a multifractal behavior implies both scale invariance and a non Gaussian behavior. One step further, the property of Extended Self-Similarity (E.S.S.) was introduced in the study of turbulent flows in the early 90s [4]. At first, it was used to increase the precision of scaling exponent estimate. It relates moments of different orders through a relative scaling behavior:

$$\mathbf{E}\varepsilon_r^q \propto (\mathbf{E}\varepsilon_r^p)^{H(p,q)}. \quad (3)$$

Scaling of the form given in (2) clearly implies E.S.S..

Several authors [5,6] have noted analogies between the scaling properties of images and the statistics of turbulent flows. Turiel *et al.* [6] used the property of E.S.S. to show that the statistics of natural images resemble those of turbulent flows. Note that the E.S.S. property does not simply reduce to a scaling property. It betrays the evolution of the probability density functions of a scale dependent quantity (locally averaged dissipation in turbulence, locally averaged contrast in natural images), denoted by ε_r here, from the larger scales to the finer. Indeed, the E.S.S. can be seen as the signature of what is called an *infinitely divisible cascade scaling*. This was first observed on 1D signals such as hot wire velocity measurements in turbulence [4] or Internet traffic flows [7].

Beyond statistical analysis, there is also a need for actual models and tools *to synthesize* processes with controllable scaling properties. To this respect, multiplicative cascades result intimately connected to multifractal processes so that they have played a key role in turbulence. A nice feature of multiplicative cascades is that their synthesis relies on an easy to implement iterative procedure. A succession of refinements and generalizations of such multiplicative cascades led to the *infinitely divisible cascades* (IDC). IDC provide us with a versatile family of non Gaussian scale invariant processes which result easy to synthesize numerically. For 1D signals (time series), IDC have given a way to the synthesis of a large family of multifractal processes with prescribed properties [8–12]. This paper aims at showing how the 1D IDC construction generalizes to N dimensions, $N \geq 1$, with again many appealing properties: scaling exponents can be prescribed in (2); the scaling range can be precisely defined; properties are observed

over a continuum in space and scale, e.g., there is no preferred scale ratio as in discrete constructions; a wide class of non Gaussian models is available. In N dimensions, geometrical features (e.g., anisotropy) can be taken into account. Focusing on the 2D case, it appears in the following that IDC models are consistent with most of known results on the statistics of natural images (see [1] for a review).

Thus, we extend the definition of infinitely divisible cascades from 1 to N ($N \geq 2$) dimensions and focus on the 2D case in particular as it provides us with a relevant statistical model for natural images. We show that 2D IDC meet the main statistical properties of a wide class of natural images. We also point out that Mumford & Gidas [13] proposed a class of very similar infinitely divisible models from a completely different point of view. We argue that the IDC approach may give a new point of view to better understand experimental observations and to compare various existing theoretical models. Finally, various other possible applications to 2D images are considered including noise modeling, texture synthesis, or solar images modeling.

2. INFINITELY DIVISIBLE CASCADES IN N DIMENSIONS

2.1. Definitions

1D IDCs [8–12] were introduced as a randomized version of the well known canonical multiplicative cascades of Mandelbrot [14]. The N -dimensional version result from a natural generalization of the 1D definition.

Let G be an infinitely divisible distribution with moment generating function $\tilde{G}(q)$ that can be written in the form $e^{-\rho(q)}$.

Let $dm(\mathbf{x}, r) = d\mathbf{x}dr/r^{N+1}$ a positive measure on the space-scale half-plane $\mathcal{P}^+ := \mathbb{R}^N \times \mathbb{R}^+$.

Let M denote an infinitely divisible, independently scattered random measure distributed by G , supported on the space-scale half-space \mathcal{P}^+ and associated to its *control measure* $dm(\mathbf{x}, r)$. The random measure M is such that

$$\mathbf{E}[\exp [qM(\mathcal{E})]] = \exp [-\rho(q)m(\mathcal{E})].$$

For all disjoint subsets \mathcal{E}_1 and \mathcal{E}_2 , $M(\mathcal{E}_1)$ and $M(\mathcal{E}_2)$ are independent random variables and $M(\mathcal{E}_1 \cup \mathcal{E}_2) = M(\mathcal{E}_1) + M(\mathcal{E}_2)$.

Definition 1.

A cone of influence $\mathcal{C}_\ell(\mathbf{x})$ is defined for every $\mathbf{x} \in \mathbb{R}^N$ as $\mathcal{C}_\ell(\mathbf{x}) = \{(\mathbf{x}', r') : \ell \leq r' \leq 1, \|\mathbf{x}' - \mathbf{x}\| < r'/2\}$ –see figure 1(a). With a given infinitely divisible randomly scattered measure M , an Infinitely Divisible Cascading noise (IDC noise) is a family of processes $Q_\ell(\mathbf{x})$ parametrized by $\ell \in (0, 1)$ of the form –see figure 1(b) for a 2D example:

$$Q_\ell(\mathbf{x}) = \frac{\exp M(\mathcal{C}_\ell(\mathbf{x}))}{\mathbf{E}[\exp M(\mathcal{C}_\ell(\mathbf{x}))]}. \quad (4)$$

Possible choices for distribution G are the Normal distribution, Poisson and compound Poisson distributions, Gamma and stable laws,... Note that the large scale in the definition of $\mathcal{C}_\ell(\mathbf{x})$ has been arbitrarily set to 1 without loss of generality. Choosing a large scale $L \neq 1$ reduces to a change of units $(\mathbf{x}, r) \rightarrow (\mathbf{x} \cdot L, r \cdot L)$.

Note that definition 1 may be extended to a more general framework by introducing some localized integration kernel² $f(\mathbf{x})$ in (4):

²This may rejoin the *random wavelet expansions* evoked in [13].

Definition 2 (with integration kernel).

$$Q_\ell(\mathbf{x}) = \frac{\exp \int f\left(\frac{\mathbf{x}-\mathbf{x}'}{r'}\right) dM(\mathbf{x}', r')}{\mathbf{E} \left[\exp \int f\left(\frac{\mathbf{x}-\mathbf{x}'}{r'}\right) dM(\mathbf{x}', r') \right]} \quad (5)$$

This definition may result useful to attenuate small scales discontinuities or to take into account some geometrical features of the images under study –see figure 2 where various choices are illustrated. This general case will be studied in greater details elsewhere [15].

2.1.1. Properties

An immediate consequence of the definition is that Q_ℓ is a *stationary positive random process* with the normalization:

$$\mathbf{E}Q_\ell = 1. \quad (6)$$

Stationarity is ensured by the invariance to translations of both the control measure $dm(\mathbf{x}, r)$ and the cone of influence $\mathcal{C}_\ell(\mathbf{x})$. This is consistent with the usual assumption that the underlying image process is stationary or equivalently invariant to translations in the image plane. The symmetry of the cone's shape (in definition 1) inflicts an *isotropic* structure as well. Moreover, Q_ℓ has a *log-infinitely divisible distribution*, that is $\log Q_\ell$ has an infinitely divisible distribution. Thus, a large number of *non Gaussian* distributions are available (Poisson, Gamma, exponential, stable...) –see figure 1(c).

One major scaling property of IDC is:

$$\mathbf{E}\varepsilon_r^q \sim r^{\tau(q)} \text{ for } r \leq 1, \quad (7)$$

where

$$\tau(q) = \rho(q) - q\rho(1) \quad (\tau(1) = 0), \quad (8)$$

for all q for which $\rho(q) = -\log \tilde{G}(q)$ is defined. The power law scaling behavior essentially roots in the choice $dm(\mathbf{x}, r) \propto 1/r^{N+1}$ ($1/r^3$ when $N = 2$). As a consequence, a *power law spectrum* $\propto k^{-(2+\tau(2))}$, $\tau(2) < 0$, is expected –see figure 1(d). Furthermore, we emphasize that not only second order statistics but higher order statistics are prescribed as well. Note that these scaling behaviors are robust in a certain sense since they are preserved when elevating Q_ℓ to some power $\alpha > 0$.

Thus, IDCs provide us with a large class of *non Gaussian scale invariant models*, with a precise control of their scaling properties and probability densities. Moreover, using (5), geometrical degrees of freedom may be introduced as well; these will be studied elsewhere [15].

2.1.2. Compound Poisson Cascades

Among the full generality of infinitely divisible cascades, the family of compound Poisson cascades [8] (CPC) plays a special role for both historical and practical reasons. CPC have been widely used to describe the statistics of turbulent flows [4] and they are easy to synthesize numerically [11]. Indeed, CPC can be built thanks to a *marked Poisson point process* $\{(\mathbf{x}_i, r_i), W_i\}$ with density $dm(\mathbf{x}, r)$. In this case, (4) takes the following form (see figure 1(a)):

$$Q_\ell(t) = \frac{\prod_{(\mathbf{x}_i, r_i) \in \mathcal{C}_\ell(\mathbf{x})} W_i}{\mathbf{E} \left[\prod_{(\mathbf{x}_i, r_i) \in \mathcal{C}_\ell(\mathbf{x})} W_i \right]}. \quad (9)$$

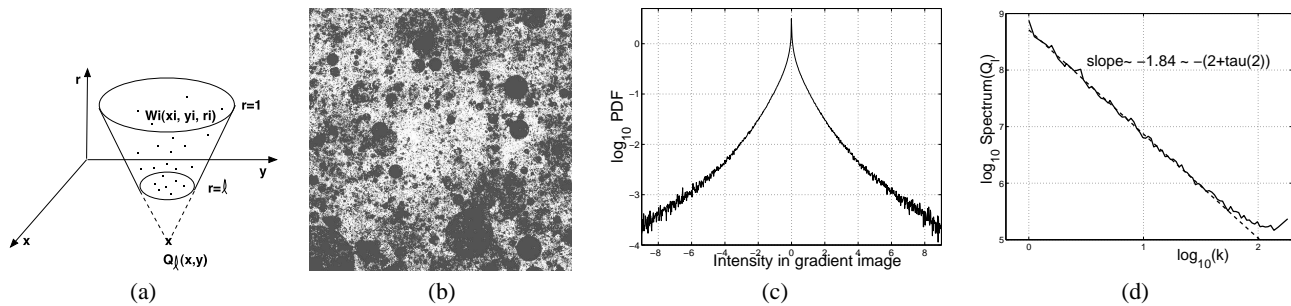


Fig. 1. (a) Space-scale cone defining $Q_\ell(\mathbf{x})$ at $\mathbf{x}(x, y)$. For a compound Poisson cascade, Q_ℓ is the product of those random multipliers $W_i(x_i, y_i, r_i)$ that belong to the cone $\mathcal{C}_\ell(\mathbf{x})$. (b) Example of a realisation of a compound Poisson cascade Q_ℓ (gray levels) with $\tau(2) \simeq -0.16$ —see definition 1. (c) Estimated (log) probability density function of the gradient image: the pdf is clearly non-Gaussian (Gaussian \Rightarrow parabola). (d) Power law spectrum of $Q_\ell(\mathbf{x})$ as a function of $k = \|\mathbf{k}\|$ over 2 decades: the observed slope is prescribed by the choice of $\tau(2)$.

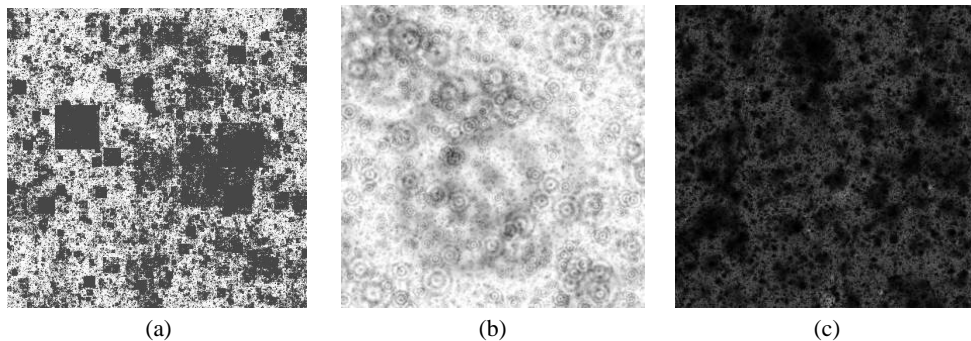


Fig. 2. (a) Example of a realisation of a compound Poisson cascade Q_ℓ (gray levels) with $\tau(2) \simeq -0.16$ using a square integration kernel $f(x, y) = 1_{[-1/2, 1/2]}(x) \cdot 1_{[-1/2, 1/2]}(y)$ —see definition 2. (b) & (c) Examples of CPC using different integration kernels.

Thus, the distribution of $Q_\ell(\mathbf{x})$ is a log-compound Poisson distribution: the Poisson distribution that comes from the point process (\mathbf{x}_i, r_i) is compound with the distribution of the i.i.d. random multipliers $W_i (> 0)$. One may as well describe CPC as the multifractal product of cylindrical pulses [8]. At this point, we emphasize that *multiplicative* models of the intensity of an image $I(\mathbf{x}) (\equiv Q_\ell(\mathbf{x}))$ are equivalent to *additive* models of the contrast $\phi(\mathbf{x}) \propto \log I(\mathbf{x}) (\equiv \log Q_\ell(\mathbf{x}))$. Algorithms for synthesis are described in [15]

3. IDC TO MODEL NATURAL IMAGES

Infinitely divisible cascades share most of the usual properties observed on image data bases [1]. Indeed, many of these features are essential attributes of multiplicative cascades.

We already mentioned that IDC exhibit a power law spectrum ($\propto k^{-(2+\tau(2))}$) and non Gaussian distributions controlled by the choice of the distribution G of Definition 1 (or equivalently by the choice of the distribution of multipliers W_i in the CPC case). But IDC obey other non trivial properties among those commonly reported on natural images.

For instance, the covariance of the intensity $I(\mathbf{x}) (\equiv Q_\ell(\mathbf{x}))$ obeys a power law. This is simply a consequence in the real space of the power spectrum in the Fourier space. But a known property

of multiplicative cascades is moreover that the covariance of the logarithm of intensity increments $\log |I(\mathbf{x}_2) - I(\mathbf{x}_1)|$ behaves as $\log \|\mathbf{x}_2 - \mathbf{x}_1\|$. Such a property was used as a starting point of the scale invariant model presented in [13]. In the framework of multiplicative cascades, this property receives some intuition since $\log \|\mathbf{x}_2 - \mathbf{x}_1\|$ is simply the average number of common ancestors $\{W_i, (\mathbf{x}_i, r_i)\}$ of the respective values of $Q_\ell(\mathbf{x}_1) \equiv I(\mathbf{x}_1)$ and $Q_\ell(\mathbf{x}_2) \equiv I(\mathbf{x}_2)$.

As far as the modeling of images is concerned, CPCs are not only a statistical model but also receive some intuitive physical interpretation. They may be interpreted as the light scattered through a random superposition of transparent cylinders of sizes $\{r_i\}$ placed at positions $\{\mathbf{x}_i\}$ and with i.i.d. random transparency W_i . This simplistic description points to the resemblance between CPC and classical approaches in image modeling since they can be compared to models where elementary objects of random sizes are distributed in space following a Poisson point process [1].

We stress again in [13] the use of infinitely divisible distributions combined to a $1/r^3$ size distribution of objects which must be connected to $dm(\mathbf{x}, r) \propto 1/r^3$ here. In [13] the notion of *clutter* is given a rather precise meaning that becomes even clearer in the framework of IDC since it can be identified to $\log(1/\ell)$. The quantity $\log(1/\ell)$ (ℓ is the resolution of an IDC) can be seen as the *depth* of the cascade since it is the average number of multipliers

W_i used to get $Q_\ell(\mathbf{x})$: the smaller ℓ , the larger the range of scales in the image and the larger the clutter. Even more precisely, the cascade Q_ℓ can be decomposed here in as many successive 'sub-cascades' as wanted by iterating the relation $Q_\ell = Q_{\ell'} \cdot Q_{\ell'}^{\ell'}$ where $Q_{\ell'}^{\ell'}$ is built using cones $\mathcal{C}_{\ell'}^{\ell'}$ in the range $\ell \leq r \leq \ell' < 1$.

Another non trivial observation receives an enlightening interpretation in the IDC framework. In [5], the link between non-Gaussian distributions and the inhomogeneity of gradients is studied by defining the coarse-grained log-contrast $\phi_N(\mathbf{x})$ of $\phi = \log(I/I_0)$ in an $N \times N$ block surrounding each point \mathbf{x} . Denote by σ_N the variance of ϕ_N for each N . Then, the normalized quantities $\phi_N(\mathbf{x})/\sigma_N$ are found to be close to identically distributed. Let \mathcal{D} denote this distribution which has zero mean and unit variance. Let ω a random variable distributed by \mathcal{D} and $W = e^\omega$. Denote by $I_N(\mathbf{x}) = \exp[\phi_N(\mathbf{x})]$. Using our notations, the result in [5] can be rewritten as:

$$\begin{aligned} \phi_N(\mathbf{x}) &\stackrel{\text{in law}}{=} \omega \cdot \sigma_N \\ &\Downarrow \\ I_N(\mathbf{x}) &\stackrel{\text{in law}}{=} W^{\sigma_N} \end{aligned} \quad (10)$$

A multiplicative hierarchy appears (in law) as the (geometrically) averaged intensity $I_N(\mathbf{x})$ over $N \times N$ pixels is concerned. Again, the framework of multiplicative cascades results relevant. This is confirmed by the fact that the same analysis performed on IDC images yields similar results with σ_N decreasing as $-\log N$. The quantity σ_N appears as a measure of the clutter at scale N , or equivalently as the depth of a multiplicative cascade. Thus, we give some precise sense to the "hidden multipliers" evoked in [1] which sound much like the Novikov's "breakwon coefficients" in turbulence [16].

The absence of occlusion effect in the IDC model may call for some comments. It should not appear as an appalling property for at least two reasons. First, edges that result from occlusion may not be the essential features within images which cause scaling. Secondly, there is no ambition to generate realistic pictures from a stochastic process realization. The purpose of such models is to capture the main common statistical features of a large class of images.

4. CONCLUSION

We have introduced the natural generalization of infinitely divisible cascades (IDC) from 1 to N dimensions. We have mainly focused on the 2D case which provides us with a wide class of non Gaussian scale invariant models for natural images. Note that scale invariance is not only meant in terms of Fourier spectrum but also of higher order statistics ($\tau(q)$ for $q \geq 2$). We have pointed out the interesting similarities between IDC and the approaches in [5, 13]. We hope that IDC will help to clarify the link between different theoretical models of images as it was the case in turbulence. We emphasize that the synthesis of IDC models results easy in many cases of interest, namely the compound Poisson cascades (CPC) family. Algorithms and MATLAB functions will be available from our web page, www.isima.fr/~chainais.

Finally, let us mention that the search for stochastic models of natural images led to the elaboration of an interesting class of processes which may serve for various applications: denoising (non Gaussian noise), the study of biological systems for vision, texture

synthesis, solar UV images, modeling of porous media radiography (2D projections of 3D sponge-like objects)... Such applications are promising directions of research under current investigation.

5. REFERENCES

- [1] A. Srivastava, A.B. Lee, E.P. Simoncelli, and S.-C. Zhu, "On advances in statistical modeling of natural images," *Journal of mathematical imaging and vision*, vol. 18, pp. 17–33, 2003.
- [2] D.J. Field, "Relations between the statistics of natural images and the response properties of cortical cells," *Journal of the Optical Society of America A*, vol. 12, pp. 2379–2394, 1987.
- [3] R. H. Riedi, "Multifractal processes," in: "Theory and applications of long range dependence", eds. Doukhan, Oppenheim and Taqqu, pp. 625–716, 2003.
- [4] U. Frisch, *Turbulence. The legacy of A. Kolmogorov*, Cambridge University Press, Cambridge, UK, 1995.
- [5] D.L. Ruderman, "The statistics of natural images," *Network: computation in neural systems*, vol. 5, pp. 517–548, 1994.
- [6] A. Turiel, G. Mato, and N. Parga, "Self-similarity properties of natural images resemble those of turbulent flows," *Physical Review Letters*, vol. 80, no. 5, pp. 1098–1101, 1998.
- [7] D. Veitch, P. Abry, P. Flandrin, and P. Chainais, "Infinitely divisible cascade analysis of network traffic data," in *IEEE Proc. of the Int. Conf. on Acoust. Speech and Sig. Proc.*, Istanbul, Turkey, 2000.
- [8] J. Barral and B. Mandelbrot, "Multiplicative products of cylindrical pulses," *Probab. Theory Relat. Fields*, vol. 124, pp. 409–430, 2002.
- [9] J.F. Muzy and E. Bacry, "Multifractal stationary random measures and multifractal random walks with log-infinitely divisible scaling laws," *Phys. Rev. E*, vol. 66, 2002.
- [10] P. Chainais, R. Riedi, and P. Abry, "Scale invariant infinitely divisible cascades," in *Int. Symp. on Physics in Signal and Image Processing, Grenoble, France*, January 2003.
- [11] P. Chainais, R. Riedi, and P. Abry, "On non scale invariant infinitely divisible cascades," *IEEE Transactions on Information Theory*, vol. 51, no. 3, pp. 1063–1083, 2005.
- [12] P. Chainais, R. Riedi, and P. Abry, "Warped infinitely divisible cascades: beyond scale invariance," *Traitement du Signal*, vol. 22, no. 1, 2005.
- [13] D. Mumford and B. Gidas, "Stochastic models for generic images," *Quarterly of applied mathematics*, vol. LIV, no. 1, pp. 85–111, 2001.
- [14] B. Mandelbrot, "Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier," *J. of Fluid Mech.*, vol. 62, pp. 331–358, 1974.
- [15] P. Chainais, "Multidimensional infinitely divisible cascades: from intermittency in turbulence to the statistics of natural images," in *preparation*.
- [16] E. A. Novikov, "Intermittency and scale-similarity in the structure of a turbulent flow," *P.M.M. Appl. Math. Mech.*, vol. 35, pp. 231–241, 1971, [voir aussi Prikl. Mat. Mekh. **35** 266-277 (1971)].