TIME-FREQUENCY-POLARIZATION ANALYSIS OF BIVARIATE SIGNALS

Julien Flamant ♮*   *Advisors: Pierre Chainais* ♮ *and Nicolas Le Bihan* †

♮ Univ. Lille, CNRS, Centrale Lille, UMR 9189 - CRIStAL, 59000 Lille, France
† CNRS/GIPSA-Lab, Grenoble, France

1. INTRODUCTION

Bivariate signals arise in many fields such as oceanography, seismology or radar. A bivariate signal (*e.g.* the electromagnetic wave field) can be either described by a time-evolving real vector $[u(t), v(t)]^T$ or equivalently by the complex signal $x(t) = u(t) + iv(t)$, where $u(t), v(t)$ are its components.

In many applications, extracting physically interpretable parameters (*e.g.* polarization properties) from bivariate signals is of interest. This Ph.D. work aims at providing novel time-frequency methods to process bivariate signals in the most general case, that is possibly multicomponent and nonstationary.

Many authors have used augmented representations to extract geometric or polarization properties for nonstationary bivariate signals, see [2–5] and references therein. A well-known method is the rotary spectrum analysis, which decomposes a bivariate signal at each frequency into clockwise and counter-clockwise rotating components.

Existing methods have in common that they rely on the use of the classical complex Fourier Transform (FT). However the classical FT of complex signals has no Hermitian symmetry, which prevents from using directly standard time-frequency tools such as the analytic signal.

We show that using the Quaternion Fourier Transform (QFT), an alternate definition of the FT, makes it possible to extend standard time-frequency representations to process bivariate signals. Fundamental theorems are derived, illustrating the relevance of the approach. It is algebraic and takes geometrical properties into account by using quaternionic algebra.

2. METHODS

The set of quaternions, denoted $\mathbb{H}$, is a four-dimensional algebra over the real numbers with basis $\{1, i, j, k\}$. The numbers i, j, k are imaginary units such that $i^2 = j^2 = k^2 = -1$. Algebraic rules are given by cyclic permutations of the fundamental relations $ij = k$ and $ij = -ji$. Quaternion multiplication is thus noncommutative. Quaternions extend
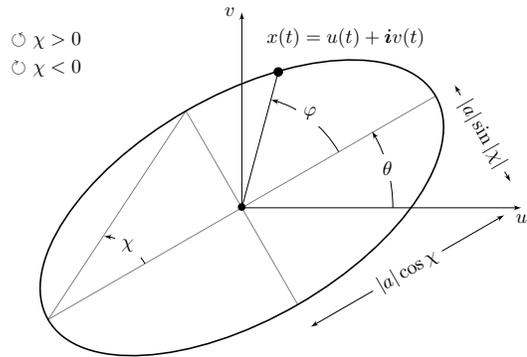
Fig. 1. Geometrical parameters of the bivariate signal x(t) identified using Eq. (4).

complex numbers. One can construct complex subfields of $\mathbb{H}$ isomorphic to $\mathbb{C}$, *e.g.* $\mathbb{C}_i = \text{span}\{1, i\}$ or $\mathbb{C}_j = \text{span}\{1, j\}$.

Let us write $x(t) = u(t) + iv(t) \in \mathbb{C}_i$ a generic bivariate signal. The key idea is now to define its Quaternion Fourier Transform (QFT) as

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) \exp(-j\omega t)\mathrm{d}t. \tag{1}$$

Note that $X(\omega) \in \mathbb{H}$. Also, terms order is important, as quaternion multiplication is noncommutative. The QFT definition is similar to the classical FT, excepted that the *axis* of the transform is j, instead of i for classical FT case. The QFT $X(\omega)$ then exhibits a particular symmetry, called the i-hermitian symmetry:

$$X(-\omega) = -iX(\omega)i. \tag{2}$$

This symmetry is at the core of the tools developed in this work.

3. RESULTS

3.1. Quaternion embedding

Symmetry (2) is fundamental. It says that the negative part of the spectrum carries no information. Thus we define a bivariate counterpart of the well-known analytic signal, hereafter termed the *quaternion embedding of a complex signal*.
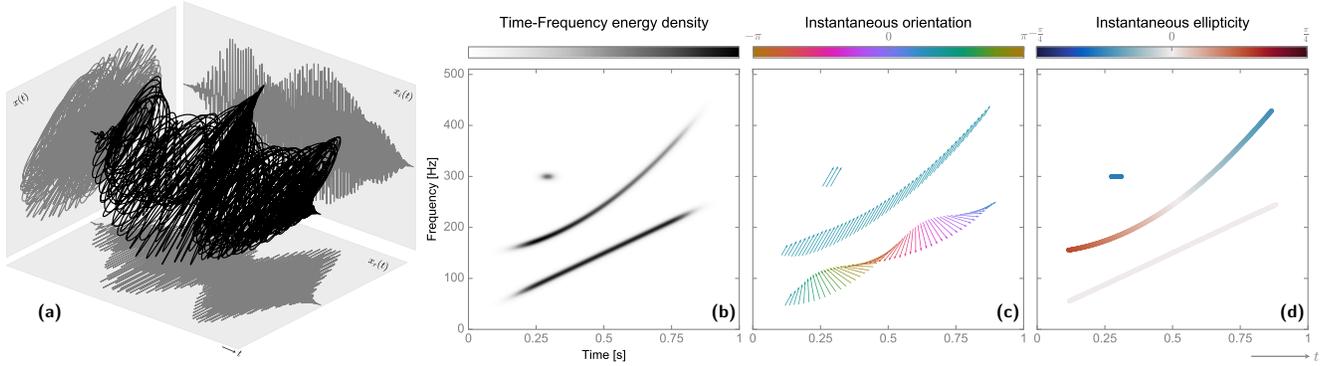
Fig. 2. (a) Synthetic signal. Polarization spectrogram: (b) Time-frequency energy density (c) instantaneous orientation (d) instantaneous ellipticity on the ridges.

It is given by

$$x_+(t) = 2 \cdot \frac{1}{2\pi} \int_0^{+\infty} X(\omega) \exp(\boldsymbol{j}\omega t) \mathrm{d}\omega, \qquad (3)$$

i.e. we have suppressed the negative frequencies spectrum of $X(\omega)$. Note that $x_+(t) \in \mathbb{H}$. We have shown that it can be factorized as [1]

$$x_+(t) = a(t) \exp[-\boldsymbol{k}\chi(t)] \exp[\boldsymbol{j}\varphi(t)], \qquad (4)$$

with $a(t) \in \mathbb{C}_{\boldsymbol{i}}$, $\chi(t) \in [-\pi/4, \pi/4]$, $\varphi(t) \in [-\pi/2, \pi/2]$. This decomposition is a quaternion analogue of the polar form of complex numbers. The projection of $x_+(t)$ onto $\mathbb{C}_{\boldsymbol{i}}$ is $x(t)$ by construction.

Fig. 1 shows the 2D trajectory of $x(t)$ for $a(t) = |a| \exp(\boldsymbol{i}\theta)$ and $\chi(t) = \chi$. The signal $x(t)$ then describes an ellipse whose scale, orientation and shape are given by $|a|$, θ and χ respectively. The phase $\varphi(t)$ gives the instantaneous position on the ellipse. These quantities may elvove with time, thus describing both the *instantaneous ellipse and phase properties* of $x(t)$ [1]. In particular $(\theta(t), \chi(t))$ can be interpreted as the instantaneous polarization properties of the bivariate signal.

3.2. Polarization spectrogram

The quaternion embedding suffers from the same drawbacks as the analytic signal, thus preventing from analyzing *multicomponent* signals. We address this issue with a Quaternion Short Term Fourier Transform (Q-STFT) based on a windowed QFT. In [1], we show that it defines an invertible time-frequency representation. In addition we prove that both energetic and polarization quantities are preserved by the Q-STFT. As a result we can meaningfully define the polarization spectrogram of a bivariate signal. A ridge analysis is also performed, validating further this new tool. Note that similar results exist for the Quaternion Continuous Wavelet Transform (Q-CWT) [1].

Fig. 2 depicts a three-component synthetic bivariate signal along with its polarization spectrogram. Each component

has various instantaneous polarization and time-frequency properties. Fig. 2b gives the time-frequency density of the signal. Fig. 2c, 2d show respectively the instantaneous orientation and ellipticity of each component. For instance, we observe that the linear chirp component exhibits a rotating orientation with null ellipticity, whereas the quadratic chirp has constant orientation and reversing ellipticity. The polarization spectrogram of the signal reveals immediatly the different features both in terms of time-frequency content and instantaneous polarization state.

4. PROSPECTS

Future work will focus on validating the approach on real-world examples and extending it towards random bivariate signals.

5. REFERENCES

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