LINEAR FILTERING OF BIVARIATE SIGNALS USING QUATERNIONS

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ABSTRACT

A new approach towards linear time-invariant (LTI) filtering of bivariate signals is proposed using a tailored quaternion Fourier transform. In the proposed framework LTI filters are naturally described by their eigenproperties providing economical, physically interpretable and straightforward filtering definitions in the frequency domain. It enables an easy design of LTI filters and a simple method for spectral synthesis of bivariate signals with prescribed frequency polarization properties. It also yields various natural decompositions of bivariate signals. Numerical experiments illustrate the approach.

***Index Terms*—** bivariate signal, polarization, LTI filtering, quaternion Fourier transform

1. INTRODUCTION

Bivariate signals appear in many fields of application such as seismology [1], optics [2], oceanography [3], etc. Usual representations involve either vectors $x(t) = (x_1(t),\ x_2(t))$ or complex signals $x(t) = x_1(t) + \boldsymbol{i}x_2(t)$. In either case, next step is the description of linear time-invariant (LTI) filters as a cornerstone of signal processing [4]. The complex representation leads to the so-called *widely linear filters* [5–8] that perform distinct operations on the signal and its conjugate. The vector representation is more common in physical sciences, *e.g.* in polarization optics [9, 10]. LTI filters are then described in the spectral domain by 2×2 complex matrices called *Jones matrices*. Such systems combine two essential physical effects which are *birefringence* and *diattenuation*; there are anisotropic optical properties. Existing approaches exhibit a number of limitations since they do not feature all the desirable properties of a complete framework to describe bivariate LTI filters: (i) the ability to manipulate bivariate signals as single algebraic objects for calculations (in contrast with *e.g.* rotary components [11]), (ii) a comfortable duality between time and frequency to define easily interpretable Fourier representations (in contrast with *e.g.* Jones matrices), (iii) a simple representation of LTI filters in terms of their main properties (*e.g.* eigenvectors/values of Jones matrices).

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We have recently proposed a powerful alternative framework based on a quaternion Fourier transform (QFT) for bivariate signal processing [12, 13] that provides a compact and elegant calculus corresponding to geometric handling of polarization states. It naturally connects usual physical quantities to well-defined mathematical (quaternionic) quantities: spectral densities, covariances, time-frequency representations, etc. Note that, with the same motivations, previous works in optics [14, 15] have proposed to use Pauli algebra rather than quaternions. However they deal with monochromatic signals only and miss a nice time-frequency duality.

The proposed framework enables an efficient description of LTI filters gathering all the desired properties mentionned above and overcomes the limitations of previous approaches. Thus LTI filters can be easily described and designed in terms of their main properties. Their effect on a signal is easy to interpret or prescribe since the proposed representation explicitly involves their eigenproperties. LTI filters consist of the action of 2 filters: an unitary filter corresponding to *birefringence*, and a Hermitian filter corresponding to *diattenuation* [10, 16]. Based on this usual decomposition, this paper describes each family using a QFT description of the frequency response. Such a framework facilitates the spectral synthesis of bivariate signals with prescribed frequency-polarization properties as well as the design of specific filters. Moreover it provides a natural path toward various useful decompositions of bivariate signals as illustrated by numerical experiments.

2. BACKGROUND

2.1. Quaternion spectral representation

Quaternions. They form a 4D algebra denoted by $\mathbb{H}$ with canonical basis $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ such that

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1,\ \boldsymbol{ijk} = -1, \tag{1}$$

and where multiplication is non-commutative, *e.g.* $\boldsymbol{ij} = -\boldsymbol{ji}$. A quaternion $q \in \mathbb{H}$ has a *scalar* or *real* part $\mathcal{S}(q) \in \mathbb{R}$ and a *vector* or *imaginary* part $\mathcal{V}(q) = q - \mathcal{S}(q) \in \operatorname{span}\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. When $\mathcal{S}(q) = 0$, q is said to be pure. Quaternions encompass complex numbers: thus standard notions such as conjugation, modulus, polar forms or rotations are readily extended. See *e.g.* [17] for a complete review on properties of quaternions.

Spectral representation. Let $x(t) = x_1(t) + \boldsymbol{i}x_2(t)$ a bivariate signal where x_1, x_2 are real signals. It is a quaternion-valued signal taking values only in $\mathbb{C}_{\boldsymbol{i}} \triangleq \operatorname{span}\{1, \boldsymbol{i}\}$. This representation of bivariate signals coupled with a tailored quaternion Fourier transform (QFT) is at the core of our framework. The QFT of $x(t)$ is defined by [12, 18]:

$$X(\nu) = \int_{-\infty}^{+\infty} x(t)e^{-\boldsymbol{j}2\pi\nu t}\mathrm{d}t \in \mathbb{H}, \tag{2}$$

which is very similar to the usual FT: the axis $\boldsymbol{i}$ of the FT has simply been replaced by $\boldsymbol{j}$. This QFT features many nice properties as well as numerical efficiency (relying on FFT) for the analysis of the spectral content of bivariate signals [12]. In particular it decomposes bivariate signals into a sum of monochromatic polarized signals. The polar form of $X(\nu)$ explicitly features meaningful polarization parameters. Moreover the Hermitian-like symmetry permits to define the polarization spectrogram, a novel time-frequency-polarization representation of bivariate signals, see [12] for details.

Remark: in full generality, dealing with random signals requires the proper statement of a spectral representation theorem based on the QFT, see Theorem 1 in [13]. In the sequel, signals are supposed either deterministic or random (second-order) stationary.

2.2. Quaternion spectral density

In [13] we have demonstrated that a *quaternion power spectral density* (PSD) can be adequatly defined. It has an elegant interpretation in terms of frequency-dependent polarization attributes and is directly related to the well-known Stokes parameters in optics [9, 13]. This quaternion PSD summarizes the second-order spectral properties of stationary bivariate signals. Its interpretation as a density is guaranteed by a generalized Parseval theorem for the QFT [12]. The quaternion PSD of a stationary bivariate signal $x(t)$ reads:

$$\Gamma_{xx}(\nu) = \underbrace{S_{0,x}(\nu)}_{\text{scalar part}} + \underbrace{\Phi_x(\nu)S_{0,x}(\nu)\boldsymbol{\mu}_x(\nu)}_{\text{vector part}} \in \mathbb{H}. \tag{3}$$

It involves two distinct quantities. Its scalar part (also called the *total PSD*)[1], $S_{0,x}(\nu) \geq 0$ is standard and characterizes the power distribution over frequencies. Its vector part describes the evolution over frequencies of the *polarization* properties of x. Given any ν, the *polarization axis* $\boldsymbol{\mu}_x(\nu)$ is a pure unit quaternion which describes the *shape* of polarization. For instance when $\boldsymbol{\mu}_x(\nu) = \pm\boldsymbol{j}$ the signal shows linear horizontal/vertical polarization at this frequency; when $\boldsymbol{\mu}_x(\nu) = \pm\boldsymbol{i}$ it has counter-clockwise/clockwise circular polarization. Note that for an arbitrary axis $\boldsymbol{\mu}$, $\pm\boldsymbol{\mu}$ denote orthogonal (in the usual sense) polarizations. The *degree of polarization* $\Phi_x(\nu) \in [0, 1]$ quantifies the amount of *polarized* components at every frequency. When $\Phi_x(\nu) = 0$ (resp. $= 1$) the signal is *unpolarized* (resp. *fully polarized*) at ν; otherwise it is *partially* polarized. The quaternion PSD expression (3) is intimately related to the Poincaré sphere [9], making the vector part of the quaternion PSD easy to interpret in terms of polarization properties [13, 19].

[1] The term "total" refers to the fact that $S_{0,x}(\nu)$ contains power contributions from the unpolarized and polarized part, see [13]. Note also that $S_{0,x}(\nu)$ is the first Stokes parameter, corresponding to the "intensity" in optics [9].

3. LINEAR TIME-INVARIANT FILTERING

The quaternion formalism provides a straightforward understanding of linear time-invariant (LTI) filters for bivariate signals. Natural interpretations and efficient design of filters are granted thanks to filtering relations directly involving eigenproperties. Remarkably described filters match well-known polarization properties of linear media: *birefringence* and *diattenuation* (also called *dichroism*) [16].

LTI filters for bivariate signals fall into two complementary classes: unitary filters and Hermitian filters. Filtering relations are given by Propositions 1 and 2 below. They are given in the spectral domain since (i) polarization properties are intrinsically related to monochromatic components and (ii) no practical convolution-like expression exists for these filters. Input and output of filters are denoted by $x(t)$ and $y(t)$, with QFTs $X(\nu)$ and $Y(\nu)$, respectively. Symmetry conditions on filter parameters ensure that $y(t)$ is a $\mathbb{C}_{\boldsymbol{i}}$-valued bivariate signal. See [19] for proofs and additional details.

3.1. Unitary filters

An unitary filter makes the polarization axis of the input rotate and leaves the degree of polarization and total PSD invariant. This is known as birefringence. Three quantities define this filter: a *birefringence axis* $\boldsymbol{\mu}(\nu)$, *birefringence angle* $\alpha(\nu)$ and *phase* $\varphi(\nu)$, both in $[0, 2\pi)$.

Proposition 1 (Unitary filter)**.** *The filtering relation is*

$$Y(\nu) = e^{\boldsymbol{\mu}(\nu)\frac{\alpha(\nu)}{2}}X(\nu)e^{\boldsymbol{j}\varphi(\nu)}, \tag{4}$$

with $\boldsymbol{\mu}(-\nu) = -\boldsymbol{i}\boldsymbol{\mu}(\nu)\boldsymbol{i}$, $\alpha(-\nu) = \alpha(\nu)$ *and* $\varphi(-\nu) = -\varphi(\nu)$. *The quaternion PSD of* $y(t)$ *is*

$$\Gamma_{yy}(\nu) = e^{\boldsymbol{\mu}(\nu)\frac{\alpha(\nu)}{2}}\Gamma_{xx}(\nu)e^{-\boldsymbol{\mu}(\nu)\frac{\alpha(\nu)}{2}} \tag{5}$$

The quantity $\varphi(\nu)$ is standard and gives the time delay corresponding to every frequency. Eq. (5) shows that such filter performs a 3D rotation of the quaternion PSD. This rotation is defined by the birefringence parameters: its angle $\alpha(\nu)$ and its axis $\boldsymbol{\mu}(\nu)$. Plugging (3) into (5) one sees that the total PSD and degree of polarization are invariant: $S_{0,y}(\nu) = S_{0,x}(\nu)$ and $\Phi_y(\nu) = \Phi_y(\nu)$, so that only the input polarization axis $\boldsymbol{\mu}_x(\nu)$ is rotated.

Let us fix ν and drop this dependence for simplicity. An unitary filter has two orthogonal fully polarized eigenpolarizations $Z_\pm$ with axes $\boldsymbol{\mu}_{z_\pm} = \pm\boldsymbol{\mu}$. Corresponding outputs are simply delayed:

$$e^{\frac{\boldsymbol{\mu}\alpha}{2}} Z_\pm e^{\boldsymbol{j}\varphi} = Z_\pm e^{\boldsymbol{j}(\varphi\pm\alpha)}. \tag{6}$$

Using a classical terminology of birefringence, Z_+ is the *fast* eigenpolarization and $Z_-(\nu)$ is the *slow* eigenpolarization. Remarkably the knowledge of these two eigenpolarizations along with associated delays completely define the filter.

3.2. Hermitian filters

A Hermitian filter affects both total PSD and polarization properties of the input. It is defined by three quantities: a *homogeneous gain* $K(\nu) \geq 0$, a *diattenuation axis* $\boldsymbol{\mu}(\nu)$ and a polarizing power $\eta(\nu) \in [0, 1]$.

Proposition 2 (Hermitian filter)**.** *The filtering relation is*

$$Y(\nu) = K(\nu)[X(\nu) - \eta(\nu)\boldsymbol{\mu}(\nu)X(\nu)\boldsymbol{j}] \tag{7}$$

with $K(-\nu) = K(\nu)$, $\eta(-\nu) = \eta(\nu)$, $\boldsymbol{\mu}(-\nu) = -\boldsymbol{i}\overline{\boldsymbol{\mu}(\nu)}\boldsymbol{i}$. *Using (3), the quaternion PSD of* $y(t)$ *is then given by (dropping* ν *dependence for convenience)*

$$\mathcal{S}(\Gamma_{yy}) = S_{0,x}K^2\left[1 + \eta^2 + 2\eta\Phi_x\langle\boldsymbol{\mu}, \boldsymbol{\mu}_x\rangle\right] \tag{8}$$

$$\mathcal{V}(\Gamma_{yy}) = S_{0,x}K^2\left[2\eta\boldsymbol{\mu} + \Phi_x[\boldsymbol{\mu}_x - \eta^2\boldsymbol{\mu}\boldsymbol{\mu}_x\boldsymbol{\mu}\right] \tag{9}$$

where $\langle\boldsymbol{\mu}_1, \boldsymbol{\mu}_2\rangle = \mathcal{S}(\boldsymbol{\mu}_1\overline{\boldsymbol{\mu}_2})$ *is the usual inner product of* $\mathbb{R}^3$.

Let us now fix ν and omit this dependence for convenience. Like unitary filters, a Hermitian filter has two orthogonal, fully polarized eigenpolarizations $Z_\pm$ with axes $\boldsymbol{\mu}_{z_\pm} = \pm\boldsymbol{\mu}$. From (7) one has

$$K[Z_\pm - \eta\boldsymbol{\mu}Z_\pm\boldsymbol{j}] = K[1 \pm \eta]Z_\pm. \tag{10}$$

Eq. (10) shows that the two eigenpolarizations are asymetrically scaled. This effect is controlled by the *polarizing power* η and is an illustration of *diattenuation*. The gain of the filter is defined as:

$$G = \frac{S_{0,y}}{S_{0,x}} = K^2\left[1 + \eta^2 + 2\eta\Phi_x\langle\boldsymbol{\mu}_x, \boldsymbol{\mu}\rangle\right] \tag{11}$$

Importantly the alignment $\langle\boldsymbol{\mu}_x, \boldsymbol{\mu}\rangle$ between input polarization axis $\boldsymbol{\mu}_x$ and diattenuation axis $\boldsymbol{\mu}$ modulates G. This effect is majored for $\eta = 1$ and for fully polarized signals $\Phi_x = 1$. When $\eta = 0$ then the gain reduces to $G = K^2$: there is no polarization-dependent effects. Remarkably, given filter parameters K and η eigenpolarizations $Z_\pm$ corresponds to maximum and minimum values of the gain G. When the filter has maximum polarizing power ($\eta = 1$) then it is a *polarizer* at this frequency. The output quaternion PSD is:

$$\Gamma_{yy} = 2S_{0,x}K^2\left(1 + \Phi_x\langle\boldsymbol{\mu}_x, \boldsymbol{\mu}\rangle\right)(1 + \boldsymbol{\mu}) \tag{12}$$

The output is fully polarized. The gain is modulated by the input degree of polarization Φ_x and the projection of the input polarization axis $\boldsymbol{\mu}_x$ onto the diattenuation axis $\boldsymbol{\mu}$.

4. APPLICATIONS

4.1. Spectral synthesis by Hermitian filtering

On one hand, the response of an Hermitian filter to an unpolarized white Gaussian noise (WGN) gives a practical identification method. The input unpolarized WGN[2] $w(t)$ has constant quaternion PSD $\Gamma_{ww}(\nu) = \sigma_0^2 \geq 0$, with σ_0^2 the noise variance [13]. Let $x(t)$ denotes the output of Hermitian filtering of $w(t)$. The output quaternion PSD is

$$\Gamma_{xx}(\nu) = \sigma_0^2K^2(\nu)[1 + \eta^2(\nu) + 2\eta(\nu)\boldsymbol{\mu}(\nu)]. \tag{13}$$

Filter parameters $\eta(\nu)$ and $\boldsymbol{\mu}(\nu)$ completely define the output polarization state and can be identified by estimating $\Gamma_{xx}(\nu)$.

On the other hand, Eq. (13) provides a way to simulate any Gaussian stationary bivariate signal with arbitrary quaternion PSD by Hermitian filtering of unpolarized WGN. It extends a well-known spectral synthesis method for scalar signals [20]. Let $\Gamma_0(\nu) = S_0(\nu)[1 + \Phi_0(\nu)\boldsymbol{\mu}_0(\nu)]$ the desired quaternion PSD. Identifying (13) with (3), one gets the parameters of filter matching the desired quaternion PSD $\Gamma_0(\nu)$

$$\begin{cases} \eta(\nu) &= \dfrac{1 - \sqrt{1 - \Phi_0^2(\nu)}}{\Phi_0(\nu)} \quad (\Phi_0(\nu) \neq 0) \\ K(\nu) &= \sqrt{S_0(\nu)/\sigma_0^2(1 + \eta^2(\nu))} \\ \boldsymbol{\mu}(\nu) &= \boldsymbol{\mu}_0(\nu) \end{cases} \tag{14}$$

and $\eta(\nu) = 0$ when $\Phi_0(\nu) = 0$. Fig. 1b gives the parameters of this signal. The total PSD corresponds to a bandpass second-order system. Importantly polarization properties are constant over frequencies: it is partially elliptically polarized, $\Phi_x = 0.8$ and $\boldsymbol{\mu}_x = -2/5\boldsymbol{i} + 3/5\boldsymbol{j} - 3/5\boldsymbol{k}$. Fig. 1a presents a simulated sequence of length $N = 1024$ obtained following a classical approach [20]; see also [19] for details. This signal is used in the next experiments. Of course, much richer signals could be similarly obtained.

4.2. Orthogonal polarizations decomposition

In many situations bivariate signals are resolved into a pair of orthogonally, fully polarized components: *e.g.* linear horizontal vertical polarization, or counter-clockwise and clockwise circular polarization. This decomposition can be generalized to any arbitrary polarization axis $\boldsymbol{\mu}$, suggesting that $x(t)$ reads

$$x(t) = x_{\boldsymbol{\mu}}^+(t) + x_{\boldsymbol{\mu}}^-(t) \tag{15}$$

where $x_{\boldsymbol{\mu}}^+(t)$ and $x_{\boldsymbol{\mu}}^-(t)$ are two orthogonally polarized components with polarization axes $\boldsymbol{\mu}(\nu)$ and $-\boldsymbol{\mu}(\nu)$, respectively.

It is possible to obtain such decompositions by LTI filtering. Since they are fully polarized, it is necessary to use Hermitian filters with polarizing power $\eta(\nu) = 1$. These

[2] Unpolarized WGN is equivalent to *proper* WGN [13].

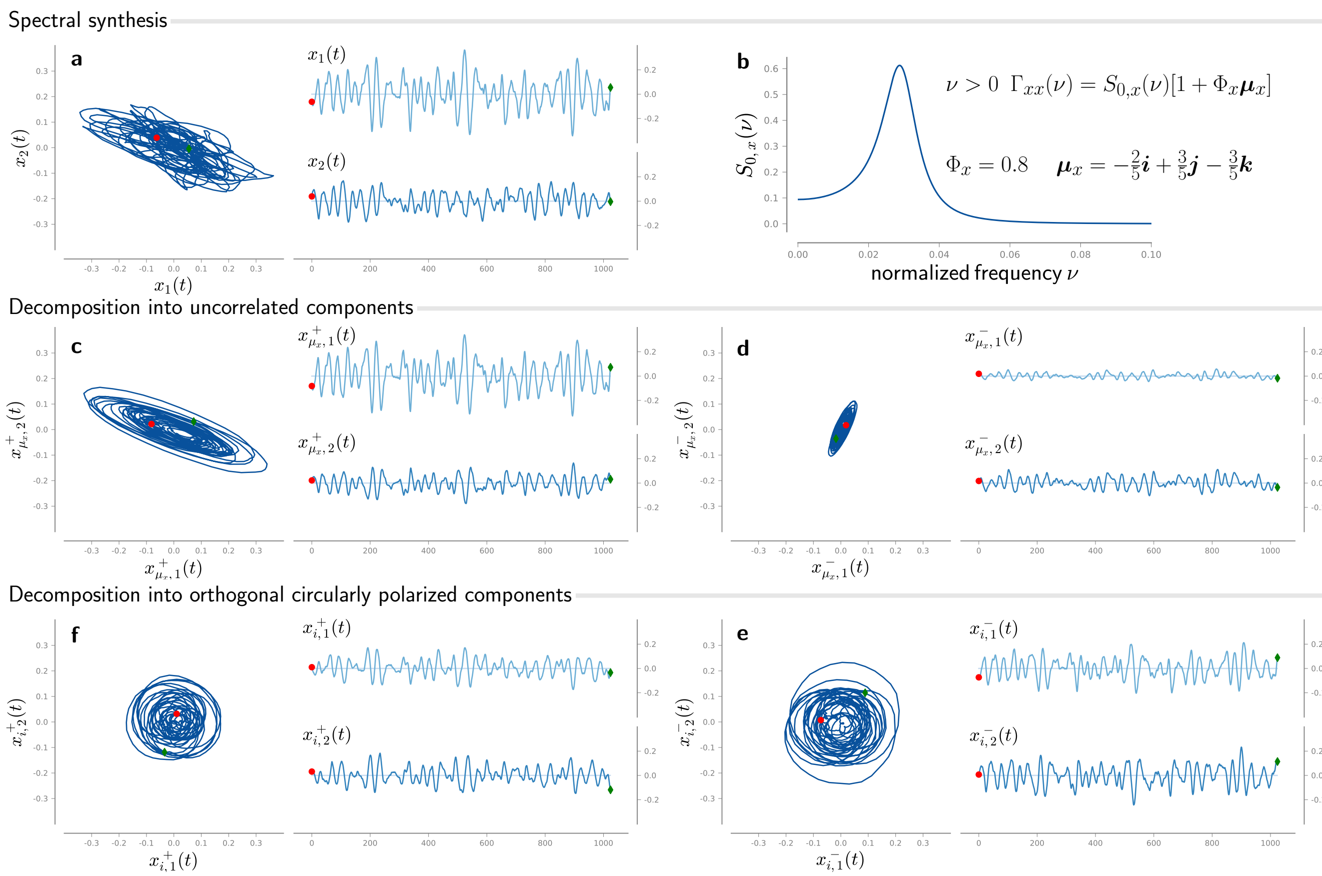

Fig. 1. (a) Partially elliptically polarized signal of Sec. 4.1 obtained by spectral synthesis, which is used in all subsequent simulations; (b) total PSD and polarization parameters used in (a); (c) and (d) uncorrelated orthogonally polarized components along $\boldsymbol{\mu}_x(\nu)$ and $-\boldsymbol{\mu}_x(\nu)$; (e) and (f) orthogonal circularly polarized components, also known as rotary components.

filters are known as *polarizers*, see Section 3.2. The definition of two orthogonal filters imposes that their diattenuation axes are $+\boldsymbol{\mu}(\nu)$ and $-\boldsymbol{\mu}(\nu)$. The condition $\forall \nu, X(\nu) = X^+_{\boldsymbol{\mu}}(\nu) + X^-_{\boldsymbol{\mu}}(\nu)$, the QFT of (15), constrains the value of the gain $K(\nu) = 1/2$ so that

$$X^+_{\boldsymbol{\mu}}(\nu) = \frac{1}{2}\left[X(\nu) - \boldsymbol{\mu}(\nu)X(\nu)\boldsymbol{j}\right] \qquad (16)$$

$$X^-_{\boldsymbol{\mu}}(\nu) = \frac{1}{2}\left[X(\nu) + \boldsymbol{\mu}(\nu)X(\nu)\boldsymbol{j}\right]. \qquad (17)$$

Note that components $x^+_{\boldsymbol{\mu}}(t)$ and $x^-_{\boldsymbol{\mu}}(t)$ are in general correlated. The two components are uncorrelated (provided that $x(t)$ is partially polarized at all frequencies) if and only if the two filters are orthogonal and $\boldsymbol{\mu}(\nu) = \boldsymbol{\mu}_x(\nu)$.

Fig. 1c and 1d present the two uncorrelated components obtained for $\boldsymbol{\mu}(\nu) = \boldsymbol{\mu}_x(\nu)$. Fig. 1e and 1f depict the two orthogonal circular components obtained with $\boldsymbol{\mu}(\nu) = \boldsymbol{i}$. Remarkably, this decomposition of the signal into counter-clockwise and clockwise circular components gives the (correlated) rotary components, widely used in both signal processing and oceanographic communities [3, 11].

5. CONCLUSION

A new approach towards LTI filtering of bivariate signals using a quaternion Fourier transform has been proposed. Recall that the QFT is efficiently computed using FFT. It enables a compact, elegant and directly interpretable definition of LTI filters that explicitly features their eigenproperties. Thus it facilitates the design of filters as the combination of an unitary and a Hermitian filter, which correspond respectively to birefringence and diattenuation effects in polarization optics. In particular, the proposed approach makes spectral synthesis straightforward in terms of frequency-dependent polarization properties. Moreover any bivariate signal can be decomposed into two uncorrelated and/or orthogonal components. Beyond these simple illustrations, this generic framework paves the way to new opportunities in the modeling, design and processing of bivariate signals, see [19] for more details, *e.g.* Wiener filtering.

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