A complete framework for linear filtering of bivariate signals
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Abstract—A complete framework for the linear time-invariant (LTI) filtering theory of bivariate signals is proposed based on a tailored quaternion Fourier transform. This framework features a direct description of LTI filters in terms of their eigenproperties enabling compact calculus and physically interpretable filtering relations in the frequency domain. The design of filters exhibiting fundamental properties of polarization optics (birefringence, di-attenuation) is straightforward. It yields an efficient spectral synthesis method and new insights on Wiener filtering for bivariate signals with prescribed frequency-dependent polarization properties. This generic framework facilitates original descriptions of bivariate signals in two components with specific geometric or statistical properties. Numerical experiments support our theoretical analysis and illustrate the relevance of the approach on synthetic data.

Index Terms—Bivariate signal, Polarization, LTI filter, Quaternion Fourier transform, Wiener denoising, Spectral synthesis, Decomposition of bivariate signals

I. INTRODUCTION

BIVARIATE signals appear in numerous physical areas such as optics [1], oceanography [2], geophysics [3], [4] or EEG analysis [5]. A bivariate signal $x(t)$ is usually resolved into orthogonal components corresponding to real-valued signals $x_1(t)$ and $x_2(t)$. Then $x(t)$ can be expressed either in vector form $x(t) = [x_1(t) \ x_2(t)]^T$ or as the complex valued signal $x(t) = x_1(t) + ix_2(t)$. Benefits of each representation have been reviewed recently [6].

Linear time-invariant (LTI) filtering theory is a cornerstone of signal processing. Its extension to the case of bivariate signals depends on the chosen representation – vector or complex form. The use of the complex representation $z(t) = x_1(t) + ix_2(t)$ leads to the concept of widely linear filtering [7], [8], [9], [10], [11], meaning that the signal $x(t)$ and its conjugate $\bar{x}(t)$ are in general filtered differently. While the use of the complex representation is often advocated for in the signal processing literature [10], [12], the use of the vector form $x(t) = [x_1(t) \ x_2(t)]^T$ is more common in physical sciences, e.g. polarization optics [13], [14]. The vector $x(t)$ is usually replaced by its analytic signal version – the so-called Jones vector. LTI filters are then represented in the spectral domain by $2 \times 2$ complex matrices called Jones matrices. These matrices describe optical elements or media with fundamental optical properties such as birefringence and diattenuation. See e.g. [15] for a review of the Jones formalism.

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axes for birefringence. For diattenuation they correspond to the minimum and maximum values of the gain of the filter. This complete framework provides a new interpretable and generic approach to standard signal processing operations such as spectral synthesis and Wiener filtering for instance. Moreover it makes nature various original descriptions of bivariate signals in two components with specific geometric or statistical properties.

This paper is organized as follows. In Section II we gather useful properties of the QFT. Based on a usual decomposition [15], [14] which separates LTI filters into unitary and Hermitian ones, Section III presents a thorough study of each family in the QFT domain. Section IV presents practical applications of those filters: usual ones (spectral synthesis, Wiener filtering) and original decompositions of bivariate signals into two components with prescribed properties. Section V gathers concluding remarks. Detailed calculations are remitted to appendices. For the sake of reproducibility, an implementation of those filters: usual ones (spectral synthesis, Wiener filtering) including various original descriptions of bivariate signals in two components with specific geometric or statistical properties. Moreover it makes natural various original descriptions of bivariate signals in two components with specific geometric or statistical properties.

As it is essential to our analysis we mention another property of quaternions. Any quaternion can be represented as a pair of complex numbers. Let \( q = q_1 + iq_2, q_1, q_2 \in \mathbb{C}_j \). The vector representation of \( q \) is the 2-dimensional complex vector \( \mathbf{q} = [q_1, q_2]^T \in \mathbb{C}^2 \). For more about quaternions, the reader is referred to dedicated textbooks e.g. [24].

B. Quaternion Fourier transform

Several Quaternion Fourier transforms have been proposed so far, see [25] for a review. We briefly survey the Quaternion Fourier Transform (QFT) first introduced in [26] and further studied in [17]. Recent works [17], [18] have demonstrated the relevance of this QFT to process bivariate signals. In particular, the QFT decomposes directly bivariate signals into a sum of polarized monochromatic signals. It also allows novel, natural and direct interpretation of polarized vibration features for bivariate signals.

A bivariate signal written as a \( \mathbb{C}_j \)-valued signal reads \( x(t) = x_1(t) + j x_2(t) \), where \( x_1, x_2 \) are real signals. Suppose for now that \( x(t) \) is deterministic. The QFT of \( x(t) \) is then

\[
X(\nu) \triangleq \int_{-\infty}^{+\infty} x(t) e^{-j2\pi\nu t} \, dt = X_1(\nu) + jX_2(\nu) \in \mathbb{H}.
\]

where \( X_1, X_2 \) are the standard Fourier transform (FT) of \( x_1, x_2 \), taken as \( \mathbb{C}_j \)-complex valued. The inverse QFT is given by

\[
x(t) = \int_{-\infty}^{+\infty} X(\nu) e^{j2\pi\nu t} \, d\nu.
\]

The QFT [3] is very similar to the usual FT where the axis \( i \) of the FT has simply been replaced by \( j \). Importantly, the exponential kernel is located on the right, a crucial point due to the noncommutative nature of the quaternion product. Eq. [5] shows that a bivariate signal \( x(t) \in \mathbb{C}_i \) has a quaternion-valued spectral description \( X(\nu) \in \mathbb{H} \). Moreover the QFT of \( \mathbb{C}_i \)-valued signals exhibits the \( i \)-Hermitian symmetry [26]

\[
X(-\nu) = X(\nu)^i.
\]

Eq. [5] illustrates that for bivariate signals negative frequencies carry no information additional to positive frequencies. In [17] we demonstrated that it permits to construct a direct bivariate counterpart of the usual analytic signal by canceling out negative frequencies of the spectrum. This first tool called the quaternion embedding of a complex signal allows identification of both instantaneous phase and polarization (i.e. geometric) properties of narrow-band bivariate signals. This approach can be extended to wideband signals using a polarization spectrogram based on a short-time QFT. See [17] for details.

For finite energy signals a generalized Parseval-Plancherel theorem gives yields two invariants:

\[
\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(\nu)|^2 d\nu,
\]

\[
\int_{-\infty}^{+\infty} \Re(x(t)jx(t)) dt = \int_{-\infty}^{+\infty} \Re(X(\nu)jX(\nu)) d\nu.
\]

Eq. [5] is classical, energy is conserved. Eq. [7] illustrates that an additional quadratic quantity of geometric nature is

\[\text{documentation available at } \text{https://bispy.readthedocs.io/}\]
C. Quaternion spectral density of bivariate signals

The QFT has two invariants (6) and (7). As a result for finite energy deterministic signals the quantities $|X(\nu)|^2$ and $X(\nu)j\bar{X}(\nu)$ summarize the second-order spectral properties of the bivariate signal $x(t)$. These quantities can be adequately combined to form a quaternion energy spectral density:

$$\Gamma_{xx}(\nu) = |X(\nu)|^2 + X(\nu)j\bar{X}(\nu). \quad (8)$$

Many signals however are random and only of finite power, which makes the spectral density definition (8) no longer applicable. Fortunately thanks to a spectral representation which makes the spectral density definition (8) no longer a point on the surface of Poincaré sphere of radius this quaternion identifies a vector of $\mathbb{R}^3$.

The scalar part of $\Gamma_{xx}(\nu)$, $S_{0,x}(\nu)$, normalizes $\Phi_{x}(\nu)$ identifies a vector in $\mathbb{R}^3$ which describes the polarization attributes of $x(t)$ at frequency $\nu$. Spherical coordinates $(2\theta, 2\chi)$ gives the orientation $\theta$ and ellipticity $\chi$ of the polarization ellipse; $\mathbf{\mu}_x(\nu)$ can also be specified in Cartesian coordinates using normalized Stokes parameters, see e.g. (14) for details. Orthogonal polarizations correspond to antipodal points on the Poincaré sphere of radius $\Phi_x = 1$: e.g. clockwise and counter-clockwise circular are orthogonal polarizations. While it may sound disturbing at first, two axes $\mathbf{\mu}_x$ and $\mathbf{\mu}_y$ correspond to orthogonal polarizations in the usual sense when they are anti-aligned $\langle \mathbf{\mu}_x, \mathbf{\mu}_y \rangle = -1$.

Relation to complex signal processing. The statistical characterization of stationary random complex signals $x(t) = u(t) + iv(t)$ has been extensively studied in the signal processing literature: see e.g. (12), (10), (27) and references therein. Second-order properties of $x(t)$ are fully given by the auto-covariance $R_{xx}(\tau) = \mathbb{E}\{x(t)x(\tau)\}$ and pseudo-covariance function $\tilde{R}_{xx}(\tau) = \mathbb{E}\{x(t)x(\tau)\}$. In the spectral domain the power spectral density (PSD) $P_{xx}(\nu) = \mathcal{F}R_{xx}$ and the complementary PSD (C-PSD) $\tilde{P}_{xx}(\nu) = \mathcal{F}\tilde{R}_{xx}$ encode the second-order structure of $x(t)$, where $\mathcal{F}$ denote the usual complex Fourier transform. Note that the PSD is real nonnegative, not necessary symmetric $P_{xx}(\nu) \geq 0$ and that the C-PSD is even and complex-valued $\tilde{P}_{xx}(\nu) = P_{xx}(\nu) \in \mathbb{C}$.

The quaternion PSD (9) can be expressed in terms of $P_{xx}(\nu)$ and $\tilde{P}_{xx}(\nu)$ as (see e.g. (10) p. 213) and (18):

$$\Gamma_{xx}(\nu) = EvP_{xx}(\nu) + iOdP_{xx}(\nu) + jRe\tilde{P}_{xx}(\nu) + kIm\tilde{P}_{xx}(\nu) \quad (10)$$

where $EvP_{xx}(\nu)$ and $OdP_{xx}(\nu)$ denote respectively the even and odd parts of $P_{xx}(\nu)$ and where $Re\tilde{P}_{xx}(\nu)$ and $Im\tilde{P}_{xx}(\nu)$ denote the real and imaginary parts of $\tilde{P}_{xx}(\nu)$, respectively. A second-order stationary signal $x(t)$ is said to be proper or second-order circular if its complementary covariance or C-PSD vanishes, i.e. if $\tilde{R}_{xx}(\tau) = 0, \forall \tau$ or $\tilde{P}_{xx}(\nu) = 0, \forall \nu$;
III. LTI FILTERING FOR BIVARIATE SIGNALS

The purpose of this section is to write a complete and clean formulation of the theory of linear-time invariant (LTI) filtering for bivariate signals within the QFT framework.

LTI filters can be classified into two categories: unitary filters and Hermitian filters. This decomposition originates from optics, where one usually separates birefringence and diattenuation effects (Hermitian) from Stokes parameters (unitary). It is often implicitly assumed that one operates at a single frequency. In contrast we provide frequency-dependent expressions for unitary and Hermitian filters to deal with generic wideband bivariate signals. It must be pointed out that in general, in the time-domain there is no simple form involving a convolution for these filters.

The quaternion representation offers a direct description of these filters in terms of birefringence and diattenuation parameters. Precisely, the use of quaternion algebra allows to write unitary and Hermitian filters in terms of eigenvectors and eigenvalues of their matrix representation. It explicitly uses the eigenpolarizations of the filter, giving a natural way to identify the parameters of each filter.

Section III-A recalls that any LTI filter can be decomposed, at each frequency, into the combination of a unitary and a Hermitian transform. Lemmas 1 and 2 give quaternion representations of such transforms. Section III-B and III-C study unitary filters and Hermitian filters, respectively. We emphasize physical and geometric interpretations of these two filters. See Appendix A for technical details.

A. Matrix and quaternion representation

In the following, time-domain (resp. frequency-domain) quantities are given in lowercase letters (resp. uppercase). Scalar quantities (in general, quaternion-valued) are denoted by standard case letters \( x, X \). Vectors are denoted by bold straight letters \( \mathbf{x}, \mathbf{X} \) and matrices are written as bold straight underlined letters \( \mathbf{m}, \mathbf{M} \). Vector and matrices are always complex \( \mathbb{C}_j \)-valued.

A generic LTI filter is described by its matrix impulse response \( \mathbf{m}(t) \in \mathbb{C}_j^{2 \times 2} \) or by its Fourier Transform (FT) \( \mathbf{M}(\nu) \in \mathbb{C}_j^{2 \times 2} \). In the frequency domain the filtering relation between bivariate signals \( x \) and \( y \) reads:

\[
\mathbf{Y}(\nu) = \mathbf{M}(\nu)\mathbf{X}(\nu). \tag{11}
\]

For each \( \nu \), Eq (11) defines a linear relation between vectors \( \mathbf{Y}(\nu) \) and \( \mathbf{X}(\nu) \). For the rest of this section we fix \( \nu \) and drop now this dependence. The polar decomposition (28) of \( \mathbf{M} \) is

\[
\mathbf{M} = \mathbf{U} \mathbf{H}, \tag{12}
\]

where \( \mathbf{U} \) is unitary and \( \mathbf{H} \) is Hermitian semi-definite positive, i.e. \( \mathbf{H}^* = \mathbf{H} \) and its eigenvalues are nonnegative. Geometrically (12) decomposes \( \mathbf{M} \) as a stretch (Hermitian semi-definite positive) followed by a rotation (unitary matrix \( \mathbf{U} \)). The polar decomposition (12) suggests to study separately two fundamental transforms, respectively unitary and Hermitian ones. Remarkably these two transforms have a direct interpretation in the quaternion representation. In particular parameters are directly related to eigenvectors and eigenvalues of each transform.

Recall the equivalence between vector and quaternion representations:

\[
\mathbf{X} = [X_1, X_2]^T \in \mathbb{C}_j^2 \iff X = X_1 + iX_2 \in \mathbb{H}, X_1, X_2 \in \mathbb{C}_j. \tag{13}
\]

Lemma 1 gives the representation of unitary transforms in the quaternion domain.

Lemma 1 (Unitary transform). Let \( \mathbf{U} \in \mathbb{C}_j^{2 \times 2} \), be a unitary matrix, i.e. such that \( \mathbf{UU}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}_2 \), where \( \mathbf{I}_2 \) is the identity matrix in \( \mathbb{C}_j^{2 \times 2} \). Then

\[
\mathbf{Y} = \mathbf{U}\mathbf{X} \iff \mathbf{Y} = e^{\mu \beta} X e^{i \varphi} \tag{14}
\]

where \( \mu^2 = -1 \), and \( \beta, \varphi \in [0, 2\pi) \).

The proof is given in Appendix A-B. The parameter \( \varphi \) is the argument of \( \det \mathbf{U} \). When \( \varphi = 0 \) the unitary matrix \( \mathbf{U} \) has unit determinant and (14) highlights the well known quaternion representation of special unitary matrices. The parameter \( \mu \) gives the eigenvectors of \( \mathbf{U} \), while \( \beta \) encodes its eigenvalues, see Appendix A-B.

Lemma 2 gives the representation of Hermitian transforms in the quaternion domain.

Lemma 2 (Hermitian transform). Let \( \mathbf{H} \in \mathbb{C}_j^{2 \times 2} \) be Hermitian positive semi-definite. Then

\[
\mathbf{Y} = \mathbf{HX} \iff \mathbf{Y} = K[X - \eta \mu Xj] \tag{15}
\]

where \( \mu^2 = -1, K \in \mathbb{R}^+ \) and \( \eta \in [0, 1] \).

The proof is given in Appendix A-C. The parameter \( \mu \) encodes the eigenvectors of \( \mathbf{H} \). Parameters \( K \) and \( \eta \) depend on, respectively, the sum and difference of eigenvalues, see Appendix A-C.

The quaternion representation allows a direct interpretation and control of each transform parameters. More importantly these key results enable efficient design of unitary and Hermitian filters, see Sections III-B and III-C below.
B. Unitary filters

A unitary filter performs a unitary transform for each frequency. Such filter only modifies the polarization axis of the input signal: the total PSD and degree of polarization are not affected. It is defined by three frequency-dependent quantities: a birefringence axis $\mu(\nu)$, a birefringence angle $\beta(\nu)$ and phase $\varphi(\nu)$. The parameter $\varphi(\nu)$ is classical and quantifies the time delay associated to each frequency. Quantities $\mu(\nu)$ and $\beta(\nu)$ model birefringence \cite{14,15}. This phenomenon is of fundamental importance in many areas e.g. for modeling polarization mode dispersion in optical fibers \cite{30,31} or shear wave splitting in seismology \cite{32,33}.

Proposition 1 gives the unitary filtering relation for bivariate signals. Relations between corresponding quaternion PSDs are given below, which permit further physical and geometric interpretations.

**Proposition 1** (Unitary filter). Let $x$ and $y$ be $\mathbb{C}_4$-valued bivariate signals with respective quaternion-valued QFTs $X$ and $Y$, corresponding to the input and output of the unitary filter, respectively. The filtering relation is

$$Y(\nu) = e^{\mu(\nu)\beta(\nu)}X(\nu)e^{\jmath\varphi(\nu)},$$

with $\mu(-\nu) = \overline{\mu(\nu)}\jmath$, $\beta(-\nu) = \beta(\nu)$ and $\varphi(-\nu) = -\varphi(\nu)$. The quaternion PSD of $y$ is

$$\Gamma_{yy}(\nu) = e^{\mu(\nu)\jmath\beta(\nu)}\Gamma_{xx}(\nu)e^{-\mu(\nu)\jmath\beta(\nu)}.$$  \hspace{1cm} (17)

**Sketch of proof.** Eq. (16) is obtained directly from Lemma 1. To obtain (17), use the correspondence described in Appendix D. Plugging (16) into the quaternion PSD definition \cite{73} yields (17).

Symmetry conditions in (16) ensure that the $\jmath$-Hermitian symmetry \cite{5} is satisfied for $Y(\nu)$ so that the output $y(t)$ remains $\mathbb{C}_4$-valued for an $\mathbb{C}_4$-valued input $x(t)$. Using (10), it would be possible by developing (17) to give the equivalent nearly linear filtering relation between usual PSD and C-PSD. However, this relation would not be as straightforward and interpretable as (17).

Plugging (9) into (17) yields

$$\Gamma_{yy}(\nu) = e^{\mu(\nu)\jmath\beta(\nu)}S_{0,0}(\nu)[1 + \Phi_x(\nu)\mu_x(\nu)]e^{-\mu(\nu)\jmath\beta(\nu)}$$

$$= S_{0,0}(\nu) + \Phi_x(\nu)e^{\mu(\nu)\jmath\beta(\nu)}\mu_x(\nu)e^{-\mu(\nu)\jmath\beta(\nu)}.$$  \hspace{1cm} (18)

Eqs. (17)–(18) show that the unitary filter performs a geometric operation: a 3D rotation of the quaternion PSD $\Gamma_{xx}(\nu)$. Birefringence affects the output polarization axis $\mu_y(\nu)$, which is given by the rotation of the input polarization axis $\mu_x(\nu)$. Birefringence axis $\mu_x(\nu)$ and angle $\beta(\nu)$ define this rotation. This geometrical operation can be visualized on the Poincaré sphere in Fig. 1. Eq. (18) highlights that the total PSD and degree of polarization are rotation invariant: $S_{0,0}(\nu) = S_{0,0}(\nu)$ and $\Phi_y(\nu) = \Phi_x(\nu)$. The output polarization axis $\mu_y(\nu)$ is given by the rotation of angle $\beta(\nu)$ of $\mu_x(\nu)$ around the axis $\mu(\nu)$.

**Eigenpolarizations.** At a given $\nu$, unitary filters have two orthogonal eigenpolarizations. These are fully polarized spectral components $Z_\pm(\nu)$ with polarization axis is $\mu_\pm(\nu) = \pm \mu(\nu)$. As result one gets

$$e^{\mu(\nu)\jmath\beta(\nu)}Z_\pm(\nu)e^{\jmath\varphi(\nu)} = Z_\pm(\nu)e^{\jmath(\varphi(\nu)\pm\beta(\nu)/2)}.$$  \hspace{1cm} (19)

Eq. (19) is another illustration of birefringence. It shows that unitary filters introduce a phase difference $\beta(\nu)$ between the fast eigenpolarization $Z_+(\nu)$ and slow eigenpolarization $Z_-(\nu)$.

Eigenpolarizations properties \cite{19} give a simple way to identify the parameters of the filter. The approach is analogous to what is done in experimental optics \cite{14}. Working with monochromatic signals of increasing frequency, one can adjust the input polarization axis such that the output polarization axis is the same. It gives immediately the birefringence axis $\mu(\nu)$. Measuring phase delays with respect to fast and slow eigenpolarizations then permits using (19) to identify birefringence angle $\beta(\nu)$ and phase $\varphi(\nu)$.

C. Hermitian filters

A Hermitian filter performs a Hermitian transform at each frequency. This second type of filter acts on both power and polarization properties of the input signal. Three frequency-dependent quantities are necessary to define a Hermitian filter: the homogeneous gain $K(\nu) \geq 0$ and two quantities related to diattenuation: the polarizing power $\eta(\nu)$ and diattenuation axis $\mu(\nu)$. When $\eta(\nu) = 0$, $K(\nu)$ has a classical interpretation as the gain of the filter. When $\eta(\nu) \neq 0$ the gain of the filter depends on the projection of the polarization axis $\mu_x(\nu)$ onto the diattenuation axis $\mu(\nu)$. In particular eigenpolarizations, which are spectral components with polarization axis $\pm \mu(\nu)$ correspond to maximum and minimum gain values.

Proposition 2 gives the Hermitian filtering relation for bivariate signals. Relations between input and output quaternion PSDs are presented. The use of (20) yields an explicit rewriting of $\Gamma_{yy}(\nu)$ in terms of input polarization properties.

**Proposition 2** (Hermitian filter). Let $x$ and $y$ be $\mathbb{C}_4$-valued bivariate signals with respective quaternion-valued QFTs $X$ and $Y$, corresponding to the input and output of the Hermitian filter, respectively. The filtering relation is

$$Y(\nu) = K(\nu)[X(\nu) - \eta(\nu)\mu(\nu)X(\nu)]j$$

with $K(-\nu) = K(\nu)$, $\eta(-\nu) = \eta(\nu)$ and $\mu(-\nu) = \overline{\mu(\nu)}\jmath$. Using (9), the quaternion PSD of $y$ is then given by (dropping $\nu$ dependence for convenience)

$$S(\Gamma_{yy}) = S_{0,0}K^2[1 + \eta^2 + 2\eta\Phi_x(\mu_x, \mu_x)]$$

$$\Gamma_{yy} = S_{0,0}K^2[2\eta\mu_x + \Phi_x(\mu_x, \mu_x)]$$

where $\langle \mu_1, \mu_2 \rangle = S(\mu_1^\ast \mu_2)$ is the usual inner product of $\mathbb{R}^3$.

**Sketch of proof.** Eq. (20) is obtained directly from Lemma 2. To obtain (21)–(22), use the correspondence described in Appendix D. Plugging (20) into the quaternion PSD definition \cite{73} with the use of (9) yields (21)–(22).
Symmetry conditions in (20) ensure that the \( i \)-Hermitian symmetry is satisfied for \( Y(\nu) \) so that \( y(t) \) remains \( \mathbb{C}_i \)-valued for a \( \mathbb{C}_i \)-valued input \( x(t) \). As with unitary filters, the equivalent widely linear relation involving the usual PSD and C-PSD could be derived by developing (21)-(22) and using (10). The resulting relation would not be as economical and directly interpretable as (21)-(22) which are provided by the QFT framework. In the sequel, we work at a fixed frequency \( \nu \) and drop explicit dependence in \( \nu \) to avoid notational clutter.

**Gain.** The power gain \( G \) of the filter is defined by

\[
G = \frac{S(\Gamma_{yy})}{S(\Gamma_{xx})} = \frac{S_{0,y}}{S_{0,x}}
\]  
(23)

Using Eq. (21) this gain becomes

\[
G = K^2[1 + \eta^2 + 2\eta \Phi_x(\mu, \mu_x)].
\]  
(24)

When \( \eta = 0 \) the power gain reduces to its usual expression \( G = K^2 \). When \( \eta \neq 0 \), the gain depends on \( K \) and \( \eta \) but most importantly, on the alignment \( (\mu, \mu_x) \) between diattenuation and input polarization axes.

**Eigenpolarizations.** Hermitian filters have two orthogonal eigenpolarizations. These are fully polarized spectral components \( Z_{\pm} \) with polarization axis \( \mu_{\pm} = \pm \mu \). From (20) one has

\[
K[Z_{\pm} - \eta \mu Z_{\pm} j] = K[1 \pm \eta] Z_{\pm}.
\]  
(25)

Eq. (25) characterizes diattenuation [14], [15]. Orthogonal eigenpolarizations have different gains; the polarizing power \( \eta \) controls the gap between respective gain values.

As with the unitary filter, eigenpolarization properties (25) give a natural way to identify filter parameters. Note first that eigenpolarizations correspond directly to maximum and minimum values of the gain \( G \) (24). Thus, finding the maximum and minimum value of the gain by changing the input polarization allows to identify directly parameters \( K, \eta, \) and \( \mu \).

Let \( G_{\text{max}} \) and \( G_{\text{min}} \) denote the maximal/minimal gain values, one has

\[
\frac{2\eta}{1 + \eta^2} = \frac{G_{\text{max}} - G_{\text{min}}}{G_{\text{max}} + G_{\text{min}}} \quad \text{and} \quad K^2 = \frac{G_{\text{max}} - G_{\text{min}}}{4\eta}.
\]  
(26)

Repeating the operation for a wide range of frequencies completes the characterization procedure.

**Identification using unpolarized WGN.** The quaternion PSD of the response of the Hermitian filter to an unpolarized (orthogonal polarization) and totally polarized, the output is a purely amplified/attenuated version of the input signal. Polarization properties are not modified.

**Maximal polarizing power \( \eta = 1 \).** The Hermitian filter is called a polarizer since the output polarization properties do not depend on the input polarization properties. Geometrically, starting from (22) the term \( \mu_x - \mu \mu_y \mu_x \) corresponds to the projection of \( \mu_x \) onto \( \mu \), up to a factor 2: the filter performs a projection onto the diattenuation axis \( \mu \). The output polarization axis is \( \mu_y = \mu \); the output is totally polarized \( \Phi_y = 1 \). The gain \( G \) quantifies how ‘close’ \( \mu_x \) is to \( \mu \):

\[
G = 2S_{0,x}K^2[1 + \Phi_x(\mu_x, \mu)].
\]  
(29)

In particular, for eigenpolarizations \( Z_{\pm} \):

\[
Y_+ = 2KZ_+ \quad \text{and} \quad Y_- = 0
\]  
(30)

meaning that when the input polarization axis is \( \mu_x = -\mu \) (orthogonal polarization) and totally polarized, the output cancels out. It illustrates how the alignment between input polarization and diattenuation axes affects the gain of the filter.

**IV. Applications**

**A. Spectral synthesis.**

We propose a new simulation method for Gaussian stationary random bivariate signals based on the filtering of a bivariate WGN. Eq. (27) shows that any stationary Gaussian bivariate signal with arbitrary spectral density can be obtained by Hermitian filtering of unpolarized WGN. This result allows to generalize a well-known approximate simulation algorithm [34] to the case of bivariate random signals. We note that an exact simulation method for bivariate signals with prescribed spectral density has been recently proposed in [35] using a circulant embedding approach.
Spectral synthesis of random bivariate signals

Let $\Gamma_0(\nu) = S_0(\nu)[1 + \Phi_0(\nu)\mu_0(\nu)]$ denote the quaternion PSD of the target signal to sample from. Let $w(t)$ be an unpolarized WGN: its quaternion PSD is constant $\Gamma_{uu}(\nu) = \sigma_0^2 \in \mathbb{R}^+$. Let $x(t)$ be the result of Hermitian filtering of $w(t)$. Adapting notations from [27] one gets

$$
\Gamma_x(\nu) = \sigma_0^2 K^2(\nu) [1 + \eta^2(\nu)] \left[ 1 + \frac{2\eta(\nu)}{1 + \eta^2(\nu)} \mu(\nu) \right].
$$

(31)

Remark that (31) is of the form (9). Identifying filter parameters to match the target quaternion PSD $\Gamma_0(\nu)$ yields the same expressions as in (28).

In practice one wants to generate a discrete, $N$-length realization of the signal $x(t)$. One starts by generating an i.i.d. unpolarized WGN sequence of length $M \geq N$ (see Appendix C). Filtering this sequence thanks to discrete implementation of (20) and keeping the first $N$ samples gives a discretized realization of the signal $x(t)$. As in the univariate setting [34], the quality of the simulation is increasing with $M$. Since the numerical implementation of the QFT involves only 2 standard FFTs (see Eq. (5)), the overall simulation procedure is computationally fast with $O(M \log M)$ operations.

Figure 2 depicts a realization of a narrow-band stationary random bivariate signal with constant polarization properties. The simulation is of length $N = 1024$ and was obtained using a $M = 10N$ length unpolarized WGN sequence. The signal is partially polarized $\Phi_x = 0.7$ and exhibits elliptical polarization axis. The power is distributed in a Gaussian-shaped fashion around normalized frequency $\nu_0 = 0.02$, see Figure 2 for details. Note that the instantaneous polarization state evolves with time. This is a feature of partial polarization for quasi-monochromatic signals with constant polarization axis.

B. Wiener denoising

Wiener filtering is an ubiquitous tool in signal processing. We show that the Wiener filter for bivariate signals has a convenient quaternion representation. It allows meaningful physical interpretations and a direct parametrization in terms of polarization parameters. We restrict our analysis to the denoising case. Our goal is to estimate a signal of interest $x(t)$ from which we have measurements $y(t)$ of the form

$$
y(t) = x(t) + w(t)
$$

(32)

where $w(t)$ is bivariate noise, independent from $x(t)$. All signals are assumed to be zero-mean, second-order stationary with known spectral densities. The Wiener filter solves the minimum-mean-square-error (MMSE) problem

$$
\min E \left\{ (\hat{x}(t) - x(t))^2 \right\}
$$

(33)

where $\hat{x}(t)$ is obtained by linear filtering of $y(t)$. Intuitively when searching for a polarized deterministic signal $x(t)$ in unpolarized noise $w(t)$, the Wiener filter should behave like a polarizer. It means that every spectral component of $y$ is projected along the polarization axis $\mu_x(\nu)$. Fortunately, this intuition is proven right by the generic expression of the Wiener filter.

Frequency dependence is omitted for convenience. The Wiener denoising filter is a Hermitian filter (see Appendix D for calculations):

$$
\hat{x} = \frac{S_{0,x}(1 - \Phi_x \Phi_y \langle \mu_x, \mu_y \rangle)}{S_{0,y}[1 - \Phi_y^2]} \left[ Y - \Phi_x \mu_x - \Phi_y \mu_y - Y j \right].
$$

(34)

Compared to (60), Eq. (34) explicitly gives the Wiener denoising filter in terms of polarization features of the signal $x$ and measurement $y$. Quantities $K(\nu), \mu(\nu), \eta(\nu)$ of Proposition 2...
can be readily identified from \(^{[34]}\). Note the use of the explicit form \(^{[3]}\) of \(\Gamma_{yy}(\nu) = \Gamma_{xx}(\nu) + \Gamma_{ww}(\nu)\) to simplify notations.

In many situations the noise \(w(t)\) can be assumed unpolarized for every frequency. Then \(\Gamma_{ww}(\nu) = \sigma^2(\nu) \in \mathbb{R}^+\) and

\[
\Gamma_{yy}(\nu) = S_{0,x}(\nu) + \sigma^2(\nu) + S_{0,z}(\nu) \Phi_x(\nu) \mu_z(\nu)
\]

\[S_{0,y}(\nu) \Phi_y(\nu) \mu_y(\nu)\]  

The polarization axis is not affected by the noise: \(\mu_y(\nu) = \mu_z(\nu) = \mu_x(\nu)\) for all \(\nu\). We introduce \(\alpha = S_{0,x}/\sigma^2\), the frequency-domain signal-to-noise ratio (SNR). The degree of polarization is \(\Phi_y(\nu) = \alpha(\nu)\Phi_x(\nu)/(1+\alpha(\nu))\). The Wiener filter \(^{[34]}\) then simplifies to

\[
\hat{X}(\nu) = \left[\frac{S_{0,x}(\nu)}{2\alpha + \alpha^2[1-\Phi_x^2]} \right] \left[ \frac{\Phi_x Y - \mu_x Y(j)}{1 + \alpha[1-\Phi_x^2]} \right] \nu \]  

(36)

The diattenuation of the filter is the polarization axis of the target \(\mu_x\). Homogeneous gain and polarizing power depend on the target degree of polarization \(\Phi_x\) and frequency-domain SNR \(\alpha\). In particular, when \(x\) is deterministic (hence totally polarized at all frequencies) then the Wiener filter reduces to

\[
\hat{X}(\nu) = \frac{S_{0,x}(\nu)}{2\alpha + \alpha^2[1-\Phi_x^2]} \left[ \nu (\nu) - \mu_x(\nu) \nu \right].
\]

(37)

Eq. (37) defines a polarizer and validates our initial intuition. Each spectral component of \(y\) is projected along the polarization axis \(\mu_x(\nu)\).

The MMSE is \(\varepsilon_{\text{opt}} = \mathbb{E} \{ |\hat{x}(t) - x(t)|^2 \} \) with \(\hat{x}(t)\) given by \(^{[34]}\). The MMSE can be rewritten as a frequency-domain integral (see Appendix B)

\[
\varepsilon_{\text{opt}} = \int_{-\infty}^{\infty} \varepsilon_{\text{opt}}(\nu) d\nu
\]

(38)

where \(\varepsilon_{\text{opt}}(\nu)\) is:

\[
\varepsilon_{\text{opt}}(\nu) = S_{0,x} \left( \frac{1-\Phi_x^2}{1-\Phi_y^2} \right) \left[ 1 - \frac{S_{0,x}}{S_{0,y}} \right] \left( 1 - \frac{1-\Phi_x^2}{1-\Phi_y^2} + \alpha^2(1-\Phi_x^2) \right)
\]

(39)

\[
= S_{0,x} \left( 1 - \frac{1-\Phi_x^2}{1-\Phi_y^2} + \alpha^2(1-\Phi_x^2) \right)
\]

(40)

Eqs (39)-(40) illustrate the dependence of the optimal error in terms of polarization properties of the signal \(x\), observation \(y\) or noise \(w\). Fixing all parameters except \(\langle \mu_x, \mu_w \rangle\) in (40), the optimal error is minimum when signal and noise exhibit orthogonal polarizations, i.e., when their polarization axes are anti-aligned \((\langle \mu_x, \mu_w \rangle = -1)\). The error is maximum when signal and noise have same polarization \(\langle \mu_x, \mu_w \rangle = 1\). Given \(\alpha\), asymmetry between minimum and maximum values is accentuated for strongly polarized signal and noise \((\Phi_x, \Phi_w \simeq 1)\). For \(\alpha \gg 1\) \(^{[40]}\) becomes \(\varepsilon_{\text{opt}}(\nu) \simeq S_{0,x}(\nu)/\alpha(\nu)\), while for \(\alpha \ll 1\) one gets \(\varepsilon_{\text{opt}}(\nu) \simeq S_{0,x}(\nu)\), as expected.

We conclude by a numerical example of Wiener filter denoising. The signal \(x(t)\) is taken as the synthesized signal of Fig. 2a. It is a partially elliptically polarized narrowband signal. Spectral density parameters are given in Fig 2b. Measurements \(y(t)\) are obtained using \(^{[32]}\) with \(w(t)\) a partially vertically polarized WGN, see Appendix C for details. Its quaternion PSD is \(\Gamma_{ww}(\nu) = \sigma^2(1 - 0.4j)\). Noise variance is adjusted so that SNR = –5 dB.

Figure 2c depicts the measurements \(y(t)\). Clearly, noise level is larger on the vertical axis on account of the partial vertical polarization of \(w(t)\). Figure 2d shows the output of the Wiener filter. The reconstruction SNR is \(10\log_{10}(||x(t)||^2/||\hat{x}(t) - x(t)||^2) = 9.92\) dB, where \(||\cdot\|_2\) is the standard 2-norm. It illustrates the good performances in recovering the original signal \(x(t)\).

C. Some decompositions of stationary bivariate signals

It is known \(^{[18], [13]}\) that the spectral density of a bivariate signal \(x(t)\) can be uniquely decomposed as the sum of unpolarized and totally polarized spectral densities:

\[
\Gamma_{xx}(\nu) = [1 - \Phi_x(\nu)] S_{0,x}(\nu) + \Phi_x(\nu) S_{0,x}(\nu)\]

(41)

\[1 + \mu_x(\nu)\]

where superscripts \(U\) and \(P\) stand respectively for unpolarized and polarized parts. The decomposition \(^{[41]}\) motivates the search for decompositions of the bivariate signal \(x(t)\) into two parts \(x_a(t)\) and \(x_b(t)\) such that

\[
x(t) = x_a(t) + x_b(t)
\]

(42)

Comparing \(^{[42]}\) with \(^{[41]}\), we search a linear filter such that \(x_a(t)\) is fully polarized along \(\mu_x(\nu)\) for every frequency. Additionally the two parts should satisfy: (i) \(x_a(t)\) has quaternion PSD \(\Gamma_{xx}^U(\nu)\); (ii) \(x_b(t)\) is unpolarized for every frequency, with quaternion PSD \(\Gamma_{xx}^P(\nu)\); (iii) \(x_a(t)\) and \(x_b(t)\) are uncorrelated. Unfortunately no such linear filter exists. Each requirement corresponds to a distinct filter: only one requirement at a time can be met.

Since unitary filters do not affect the degree of polarization or are not able to decorrelate two signals, it is necessary to use a Hermitian filter. Moreover since we search for \(x_a(t)\) fully polarized along \(\mu_x(\nu)\), one has to use a polarizer along the polarization axis of \(x(t)\):

\[
X_a(\nu) = K(\nu) (X(\nu) - \mu_x(\nu) X(\nu)j),
\]

(43)

\[
X_b(\nu) = X(\nu) - X_a(\nu)
\]

(44)

The second component \(x_b(t)\) is such that \(^{[42]}\) holds. Note that in \(^{[43]}\) the gain \(K(\nu)\) is not fixed. Requirements (i), (ii) or (iii) correspond to distinct values of this gain. Stated differently, \(K(\nu)\) rules the nature of the decomposition \(^{[42]}\).

Table I summarizes expressions of the gain and quaternion PSDs of \(x_a(t)\) and \(x_b(t)\) for requirements (i), (ii) and (iii). In addition correlation properties of the two components are given. To meet (i) the gain \(K(\nu)\) is adjusted thanks to \(^{[41]}\) such that \(\Gamma_{x_b,x_a}(\nu) = \Gamma_{xx}^U(\nu)\). However \(x_b(t)\) is partially polarized and components are correlated. For (ii) starting from \(^{[44]}\) and using \(^{[22]}\) with \(\mu(\nu) = -\mu_x(\nu)\) one computes the vector part of \(\Gamma_{x_a,x_b}(\nu)\). Then the gain \(K(\nu)\) is obtained by imposing \(\Phi_x(\nu) = 0\) for every \(\nu\). Fortunately the corresponding expression for \(K(\nu)\) yields \(\Gamma_{x_a,x_b}(\nu) = \Gamma_{xx}^U(\nu)\). The first component
TABLE I
DIFFERENT DECOMPOSITIONS OBTAINED BY CHANGING THE HOMOGENEOUS GAIN $K(\nu)$.

<table>
<thead>
<tr>
<th>$K(\nu)$</th>
<th>$\Gamma_{x_a,x_a}(\nu)$</th>
<th>$\Gamma_{x_b,x_b}(\nu)$</th>
<th>correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$\frac{\Phi_{x}(\nu)}{2(1 + \Phi_{x}(\nu))}$</td>
<td>$S_{0,\nu}(\nu)\Phi_{x}(\nu)[1 + \mu_{x}(\nu)]$</td>
<td>correlated</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(ii)</td>
<td>$1 - \frac{\Phi_{x}(\nu)}{\Phi_{x}(\nu) + 1 - \sqrt{1 - \Phi_{x}(\nu)^2}}$</td>
<td>$2S_{0,\nu}(\nu)K^2(\nu)[1 + \Phi_{x}(\nu)][1 + \mu_{x}(\nu)]$</td>
<td>$S_{0,\nu}(\nu)[1 - \Phi_{x}(\nu)]$</td>
</tr>
<tr>
<td>(iii)</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{S_{0,\nu}(\nu)}{[1 + \Phi_{x}(\nu)][1 + \mu_{x}(\nu)]}$</td>
<td>$\frac{S_{0,\nu}(\nu)}{[1 - \Phi_{x}(\nu)][1 - \mu_{x}(\nu)]}$</td>
</tr>
</tbody>
</table>

$x_a(t)$ is fully polarized like $x(t)$, but has weaker intensity than that of $\Gamma_{x_a}(\nu)$. Components are also correlated. Finally (iii) is fulfilled by enforcing decorrelation between $x_a(t)$ and $x_b(t)$. See Appendix D for technical details. Importantly $x_a(t)$ and $x_b(t)$ are both fully polarized with orthogonal polarization axes. Respectively intensities are controlled by the degree of polarization $\Phi_{x}(\nu)$. $\Gamma_{x_a}(\nu)$ illustrates decompositions (ii) and (iii) on the synthesized signal of Fig. 3. Decomposition (i) is not presented as it is similar to (iii), except that $x_b(t)$ is only (strongly) partially polarized.

Taking another polarization axis in (43)-(44) will not enable satisfying requirements (i)-(ii)-(iii). Indeed the filter corresponding to (ii) and defined in Table I is the unique depolarizer of $x(t)$, i.e. the only filter that outputs an unpolarized signal from a partially polarized input ($\Phi_x < 1$). Moreover the unique linear filter producing decorrelated signals for $x_a(t)$ and $x_b(t)$ is the one defined by (iii) in Table I.

This discussion answers an important and natural question. Since the decomposition (41) holds, is it possible to decompose by linear filtering any bivariate signal into uncorrelated unpolarized and polarized components? Unfortunately the answer is negative. However, this hypothetical decomposition can still be used as a synthesis tool, as already shown [18]. Moreover in practical situations where such a decomposition may be needed, one can choose the appropriate filter according to the desired requirement (i), (ii) or (iii).

V. CONCLUSION

This paper provides a complete and powerful framework for linear time-invariant filtering of bivariate signals. The proposed framework yields a direct description of filtering in terms of physical quantities borrowed from polarization optics. Our formalism reveals the specificity of bivariate signals and is crucial to the physical understanding of even basic operations such as linear filtering. The natural expression of each filter directly in terms of eigenproperties and relevant physical parameters simplifies modeling, design, calculations and interpretations. By studying in detail the two types of filters called unitary and Hermitian filters, we have also been able to give strong physical interpretations in terms of birefringence or diattenuation effects.

We have emphasized the relevance of our work on three fundamental applications of signal processing. A spectral synthesis method to simulate any Gaussian stationary random bivariate signal with desired spectral and polarization properties has been presented. It has been shown that the Wiener denoising problem can be efficiently designed in the quaternion domain, leading to new interpretations for the bivariate case. Original decompositions of bivariate signals into two parts with specific properties have been studied. Our approach paves the way to further developments in estimation, simulation and modelling of bivariate signals. The approach is numerically efficient and relies on the use of FFT. An open-source implementation of the presented framework will be soon available in the Python companion package BiSPy.

APPENDIX A
LINEAR ALGEBRA AND QUATERNION EQUIVALENCE

A. Matrix-vector and quaternion operations

Eq. (13) shows that quaternions can be represented as complex $\mathbb{C}_2$-vectors. Let $X = [X_1, X_2]^T$ and $Y = [Y_1, Y_2]^T$ complex $\mathbb{C}_2$-vectors corresponding to quaternions $X$ and $Y$. Let $M$ denote an arbitrary complex 2-by-2 matrix. The matrix-vector relation $Y = MX$ describes an arbitrary linear transform of $\mathbb{C}_2^2$.

To obtain the corresponding relation between quaternions $Y$ and $X$, write explicitly the matrix-vector relation

$$
\begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix} =
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} =
\begin{bmatrix}
aX_1 + bX_2 \\
cX_1 + dX_2
\end{bmatrix}
$$

(45)

where $a, b, c, d \in \mathbb{C}_2$. Using (13) and that for any $q = q_1 + iq_2 \in \mathbb{H}$, $q_1, q_2 \in \mathbb{C}_2$ one has $q_1 = (q + \overline{q})/2$ and $iq_2 = (q - \overline{q})/2$:

$$
Y = Y_1 + iY_2 = aX_1 + bX_2 + i(cX_1 + dX_2)
= \frac{1}{2} \left( a - bi + ic - idi \right) X
- \frac{1}{2} \left( a + bi + ic + idi \right) jXj.
$$

(46)

Eq. (46) is the quaternion domain representation of a generic linear transform of vectors of $\mathbb{C}_2^2$.

*Documentation available at [https://bipy.readthedocs.io/](https://bipy.readthedocs.io/)*
Decomposition (ii)

\[ \begin{align*}
\dot{a}, \dot{a}, 1(t) \\
\dot{b}, \dot{a}, 2(t)
\end{align*} \]

Decomposition (iii)

\[ \begin{align*}
\dot{a}, \dot{a}, 1(t) \\
\dot{b}, \dot{a}, 2(t)
\end{align*} \]

Fig. 3. Decompositions (ii) and (iii) of the bivariate signal of Fig. 2a. See Table I for expressions. a: polarized part and b: unpolarized part of decomposition (ii). Components are correlated. c and d: uncorrelated, orthogonal polarized parts of the original signal obtained thanks to decomposition (iii).

B. Unitary transforms

Consider \( U(2) = \{ U \in \mathbb{C}_j^{2 \times 2} \text{ s.t. } UU^* = U^*U = I_2 \} \) the set of unitary matrices of \( \mathbb{C}_j^{2 \times 2} \) and denote by the subset of unitary matrices with unit determinant as \( SU(2) = \{ U \in U(2) \text{ s.t. } \det U = 1 \} \). Remark that any \( U \in U(2) \) can be written as \( U = \hat{U} \det(U) \) where \( \hat{U} \in SU(2) \) and \( \det U = \exp(j\varphi) \in \mathbb{C}_j \).

Using notations from [45], the matrix \( \hat{U} \) is characterized by \( d = \bar{a}, c = -\bar{b} \) and \( |a|^2 + |b|^2 = 1 \). Thus [46] simplifies as

\[ Y = (a - bi)X = \exp(\mu \beta)X. \]  

(47)

Since \( |a|^2 + |b|^2 = 1 \), \( a - bi \) is a unit quaternion which can be reparameterized in polar form by its axis \( \mu \) and angle \( \beta \) such that

\[ \begin{align*}
\mu &= -i \text{Re} b + j \text{Im} j a + k \text{Im} j b, \\
\beta &= \arccos \text{Re} a
\end{align*} \]

(48)

(49)

Back to \( U \in U(2) \), remark that

\[ Y = UX = \hat{U} \left[ X_1 e^{j\varphi} \right], \]

(50)

so that replacing \( X \) by the quaternion \( X e^{j\varphi} \) in (47) yields,

\[ \text{For } \hat{U} \in U(2), Y = UX \iff Y = e^{\mu \theta} X e^{j\varphi}. \]

(51)

C. Hermitian transforms

Let \( H \) be Hermitian, i.e., such that \( H^\dagger = H \). Using notations from [45] one has \( a, d \in \mathbb{R} \) and \( c = -\bar{b} \in \mathbb{C}_j \). Positive semidefiniteness is given by Sylvester Criterion: \( a, d \geq 0 \), \( ad - |b|^2 \geq 0 \), which also implies that \( d \geq 0 \). Eq. (46) becomes

\[ Y = \frac{1}{2} \left( a + d \right) X - \frac{1}{2} \left( 2bk + (a - d)j \right) Xj \]

(52)

which can be reparameterized such as

\[ K = \frac{a + d}{2} \in \mathbb{R}^+ \]

(53)

\[ \mu = \frac{(a - d)j + 2bk}{\sqrt{(a - d)^2 + 4|b|^2}^{1/2}}, \mu^2 = -1 \]

(54)

\[ \eta = \frac{(a - d)^2 + 4|b|^2}^{1/2}{a + d} \in [0, 1] \]

(55)

Respective domains of \( K, \mu, \eta \) ensure that the change of variable defines a valid one-to-one mapping. Finally, the output relation reads

\[ Y = K \left( X - \eta \mu Xj \right) . \]

(56)

Parameters \( K \) and \( \eta \) can be expressed in terms of eigenvalues \( \lambda_1, \lambda_2 \) \((\lambda_1 \geq \lambda_2 \geq 0)\) of the matrix \( \mathbf{M} \):

\[ K = \frac{\lambda_1 + \lambda_2}{2} \quad \eta = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \]

(57)

APPENDIX B

WIENER FILTER DERIVATION

We keep notations from Section [IV-B]. Let \( y(t), \hat{x}(t), x(t) \) denote vector representations of \( \mathbb{C}_j \)-valued bivariate signals \( y(t), \hat{x}(t) \) and \( x(t) \). Remark that \( \{\} \) is equivalent to its vector form:

\[ \min \mathbb{E} \left\{ \| \hat{x}(t) - x(t) \|^2 \right\}, \]

(58)

where \( \| \cdot \| \) is the Euclidean norm of \( \mathbb{C}_j^2 \). The solution to 58 in the Fourier domain is well known [10]

\[ \hat{X}(\nu) = P_{xy}(\nu) P_{yx}^{-1}(\nu) Y(\nu) \]

(59)

where \( P_{xy}(\nu), P_{yx}(\nu) \) are the usual \( \mathbb{C}_j \)-valued (cross-) spectral density matrices of \( x(t), y(t) \), respectively. The Wiener filter for the denoising problem [32] is

\[ \hat{X}(\nu) = P_{xy}(\nu) P_{yx}^{-1}(\nu) Y(\nu) \]

(60)
such that (ν) is obtained from Y(ν) by 2 successive Hermitian filters, since spectral density matrices are Hermitian – and so are their sum and inverse. Introducing an intermediate variable Z one gets

\[ Z(ν) = P_{yy}^{-1}(ν)Y(ν) \]  
\[ \hat{X}(ν) = P_{xx}(ν)Z(ν) \]

Quaternions equivalents are readily obtained using [46] and definitions of matrix spectral densities in terms of Stokes parameters \( S_i \), \( i = 0, 1, 2, 3 \) [10, p. 214]:

\[ Z(ν) = 2 \left[ (1 - \Phi_0^2(ν))S_{0,0}(ν) \right]^{-1} \times (Y(ν) + \Phi_y(ν)\mu_y(ν)Y(ν)j) \]  
\[ \hat{X}(ν) = 2^{-1}S_{0,x}(ν) (Z(ν) - \mu_x(ν)\Phi_x(ν)Z(ν)j) \]

since Stokes parameters and polarization axis are related like \[ S_0\Phi\mu = iS_3 + jS_1 + kS_2. \] Plugging (63) into (64) and reorganizing terms yields to the general Wiener filter expression (34). To obtain the error expression remark that \[ \text{Eq. (60) shows that } \sqrt{\nu} = \sqrt{\nu} \]

\[ \text{valid. Instead it has to be replaced with the spectral representation theorem [18, Theorem 1] which states for harmonizable signals } x(t) \text{ there exist quaternion-valued spectral increments } \mathrm{d}X(ν) \text{ such that } \]

\[ x(t) = \int_{-\infty}^{\infty} \mathrm{d}X(ν)e^{j2\piνt}, \]

the equality being in the mean-square sense. Then one defines the quaternion PSD \( \Gamma_{xx}(ν) \) accordingly [18] as

\[ \Gamma_{xx}(ν)\mathrm{d}ν = \mathbb{E}\left\{ |\mathrm{d}X(ν)|^2 \right\} + \mathbb{E}\left\{ \mathrm{d}X(ν)j\mathrm{d}Y(ν) \right\} \]

where \( \mathbb{E}\{ \cdot \} \) denotes the mathematical expectation.

Let \( x(t) \) and \( y(t) \) be two jointly stationary bivariate signals. These signals are uncorrelated [18] if and only if, for all \( ν \)

\[ \mathbb{E}\left\{ \mathrm{d}X(ν)\mathrm{d}Y(ν) \right\} = \mathbb{E}\left\{ \mathrm{d}X(ν)j\mathrm{d}Y(ν) \right\} = 0. \]

This is the quaternion equivalent to saying that the cross-spectral density matrix is zero: \( \mathbf{P}_{xy}(ν) = 0 \).

**REFERENCES**


