

# Multifractal Random Walks as Fractional Wiener Integrals

Patrice Abry, *Senior Member, IEEE*, Pierre Chainais, *Member, IEEE*, Laure Coutin, and Vlasos Pipiras

**Abstract**—Multifractal random walks are defined as integrals of infinitely divisible stationary multifractal cascades with respect to fractional Brownian motion. Their key properties are studied, such as finiteness of moments and scaling, with respect to the chosen values of the self-similarity and infinite divisibility parameters. The range of these parameters is larger than that considered previously in the literature, and the cases of both exact and nonexact scale invariance are considered. Special attention is paid to various types of definitions of multifractal random walks. The resulting random walks are of interest in modeling multifractal processes whose marginals exhibit stationarity and symmetry.

**Index Terms**—Fractional Brownian motion, fractional Wiener integrals, infinitely divisible cascades, multifractal random walks, multifractals, scaling properties.

## I. INTRODUCTION

**R**ANDOM multifractal measures and processes (or simply, multifractals) have been used to model natural and man-made phenomena in a variety of fields, ranging from turbulence in hydrodynamics, DNA sequences in genetics, rainfall in geophysics, to stock prices in finance, teletraffic in the Internet. Multifractals are described in at least two complementary ways. On one hand, the so-called multifractal spectrum functions are used to describe (e.g., Hausdorff) dimensions of singularity exponents of multifractals. On the other hand, the so-called partition functions are used to describe scaling properties of moments of multifractals. Multifractal formalism allows one to relate the two approaches (or the two sets of functions) through the Legendre transformation. See, for example, a review article by Riedi [1] and references therein for more information on multifractals.

Much of the initial effort to construct random multifractals was directed to multifractal random measures which are non-negative. Examples are the celebrated multiplicative (binomial)

Manuscript received May 05, 2007; revised March 02, 2009. Current version published July 15, 2009. The work of P. Abry was supported by the Foundation Del Duca, Institut de France, 2007 grant. The work of V. Pipiras was supported in part by the National Science Foundation (NSF) under Grant DMS-0505628. This work was undertaken when V. Pipiras was visiting P. Abry and the Laboratoire de Physique at ENS, Lyon, France, under the support of a CNRS Visiting Researcher position.

P. Abry is with CNRS, ENS Lyon Physics Department, 69364 Lyon cedex 7, France (e-mail: patrice.abry@ens-lyon.fr).

P. Chainais is with University Clermont-Ferrand II, LIMOS UMR CNRS 6358, 63173 Aubière, France (e-mail: pchainai@isima.fr).

L. Coutin is with the University Paris Descartes, MAP5, Paris, France (e-mail: Laure.coutin@math-info.univ-paris5.fr).

V. Pipiras is with the Department of Statistics and Operations Research, University of North Carolina, Chapel Hill, NC 27599 USA (e-mail: pipiras@email.unc.edu).

Communicated by U. Mitra, Associate Editor At Large.

Color versions of Figures 1, 3, and 4 in this paper are available online at <http://ieeexplore.ieee.org>.

Digital Object Identifier 10.1109/TIT.2009.2023708

cascades of Mandelbrot [2], Kahane and Peyriere [3] or their more recent generalizations to compound Poisson cascades of Barral and Mandelbrot [4], to (log-)infinitely divisible multifractal measures of Bacry and Muzy [5], [6], Schmitt and Marsan [7], Schmitt [8], and to non-scale-invariant infinitely divisible cascades of Chainais, Riedi, and Abry [9], [10]. See also Mannersalo, Norros, and Riedi [11], Schertzer and Lovejoy [12] for other constructions of multifractals. In many applications of multifractals, however, we need multifractal processes that take both positive and negative values, and that have stationary increments. Several models of multifractal processes were proposed to address this need.

A common way (see, for example, Mandelbrot [13]) to construct a multifractal process is through

$$Z(t) = Y(X([0, t])), \quad t \in \mathbb{R} \quad (1)$$

where  $X$  is a multifractal random measure and  $Y$  is a self-similar process with stationary increments, independent of  $X$ . For more information on self-similar processes, see, for example, Samorodnitsky and Taqqu [14, Sec. 7], Embrechts and Maejima [15]. A typical choice for  $Y$  is fractional Brownian motion (fBm)  $B_H$ ,  $H \in (0, 1)$ , which is the only Gaussian  $H$ -self-similar process with stationary increments. In this case, the multifractal process (1) is also known as fBm in multifractal time, following Mandelbrot's terminology (cf. Mandelbrot [13]). An advantage of working with processes (1) is that their multifractal properties are easy to deduce from those of  $Y$  and  $X$ .

Another suggested way to construct a multifractal process is through stochastic integration as

$$Z(t) = \int_0^t Q(u) dY(u), \quad t \in \mathbb{R} \quad (2)$$

where  $Q$  is a suitable stationary nonnegative multifractal noise (for example, such as in Barral and Mandelbrot [4]) and  $Y$  is as in (1), independent of  $Q$ . Multifractal processes (2) are generally known as multifractal random walks (MRWs, in short). The case of Brownian motion (Bm)  $Y = B$  was studied and is quite easy to deal with because (2) is conditionally (on  $Q$ ) Gaussian and the moments of  $Z(t)$  can be expressed as

$$E|Z(t)|^q = E|\mathcal{N}(0, 1)|^q \cdot E \left( \int_0^t Q(u)^2 du \right)^{q/2} \quad (3)$$

where  $E$  stands for the mathematical expectation (see, for example, Bacry and Muzy [6, p. 460]). Basic interests in introducing (2) are that their multifractal properties are generally different from those of (1) (thus contributing to the collection of available multifractal models), that their generating structure is

quite different from (1) (thus offering potential alternative generating schemes of multifractal processes) and that their sample paths appear visually different from (1). In particular, regarding their generating structures,  $Q(u)$  can be thought as determining instantaneous variance of a random system (2). Since  $Q(u)$  is a multifractal noise, the variability fluctuates with bursts of activity associated with high values of  $Q(u)$ . On the other hand, the activity of a random system (1) is thought to be governed by a multifractal time  $X(t)$ . Sudden increases in  $X(t)$  are now associated with faster activity of the system (1). (See also Figs. 3 and 4 later, and Section IV-F.)

In this work, we study processes (2) with fBm  $Y = B_H$  and refer to them as MRWs with s fractional Wiener integrals or simply MRWs if there is no confusion. The case of fBm is much more complex than that of Bm because fractional Wiener integrals (integrals of deterministic functions with respect to fBm, as in (2) conditional on  $Q$ ) are more involved than usual Wiener integrals. These integrals, especially when integrands are random, have been studied in depth only recently (see Carmona, Coutin, and Montseny [16], Decreusfond and Üstünel [17], Gripenberg and Norros [18], Pipiras and Taqqu [19]). We suppose that  $Q$  in (2) is an infinitely divisible cascading (IDC) noise or simply an infinitely divisible cascade as introduced in Muzy and Bacry [5], Chainais *et al.* [9]. This appears to be the most general class of stationary multifractal noise processes up to date.

When working with fractional integrals, it is convenient to use a different parametrization for fBm. We set

$$\kappa = H - \frac{1}{2} \quad (4)$$

and write  $B^\kappa = B_H$ . Observe that the values 0,  $(-1/2, 0)$ , and  $(0, 1/2)$  of  $\kappa$  are now associated with the values  $1/2$ ,  $(0, 1/2)$ , and  $(1/2, 1)$  of  $H$ .

The IDC noise  $Q$  is viewed as a limit of a family of processes  $Q_r$  when  $r \rightarrow 0$  and is characterized by a function  $\varphi(q)$  (cf. Section II-A for more details). MRW is defined as fractional Wiener integral with  $r \rightarrow 0$  (cf. (43)).

MRWs as fractional Wiener integrals are first discussed in passing by Muzy and Bacry [5], Bacry, Delour, and Muzy [20]. More recently, they are studied in greater depth by Ludeña [21] where they are called Multifractal Fractional Random Walks. Despite some overlap, the focus of this work is quite different from Ludeña [21]. First, we take a closer look at the definition of MRWs. These objects were defined in previous works when

$$2\kappa + c\varphi(2) > 0 \quad \Leftrightarrow \quad \kappa > -\frac{c\varphi(2)}{2} =: \kappa_c \quad (5)$$

where  $\varphi(q)$  is the function mentioned above and  $c$  is a constant entering the construction of an IDC noise. (Since  $\varphi(2) < 0$ , (5) implicitly assumes that  $\kappa > 0$  and  $\kappa_c < 1/2$ .) We study here also the case when (5) does not hold, that is

$$\kappa \leq \kappa_c \quad (6)$$

for both  $\kappa > 0$  and  $\kappa < 0$ . We suggest that, in case (6), a suitable normalization should be used for (2) with  $Q_r$  replacing  $Q$

as  $r \rightarrow 0$ . This is supported by proving convergence of moments for the expected range of parameters, and by empirical evidence in simulations. Somewhat surprisingly, the inequality (6) is even simpler to deal with in moment calculations, and scaling properties of MRWs for  $\kappa \leq \kappa_c$  essentially correspond to those of MRWs defined through the usual Bm. Second, we provide natural conditions on finiteness of (even order  $2p$ ) moments of MRWs, which were conjectured by Ludeña [21], namely

$$\left(\kappa + \frac{1}{2}\right)2p - 1 + c\varphi(2p) > 0 \quad (7)$$

under the condition (5). This is achieved through a direct approach (covering both (5) and (6) for  $\kappa > 0$ ) whereas Ludeña [21] used power counting methods leading to stronger conditions. Third, we illustrate a number of our results (such as convergence, scaling properties) through numerical simulations. In particular, this explores the practical relevance of some theoretical results that are asymptotic in nature, and raises other questions such as oversampling in simulation of MRWs. Giving a proper credit, it should be said, however, that Ludeña [21] addresses an important question of asymptotic behavior of  $p$ -variations of MRWs. This is not the focus in this work.

The rest of the paper is organized as follows. The next section, Section II, contains some preliminaries on infinitely divisible cascades and fractional Wiener integrals. Definitions of MRWs are studied in Section III. Properties of MRWs can be found in Section IV. The proofs of several auxiliary results are moved to Section V.

## II. PRELIMINARIES

### A. Infinitely Divisible Cascading (IDC) Noise

We recall here the definition of IDC noise from Chainais *et al.* [9] that will be used for integrands  $Q$  in the definition of (2).

*Infinitely Divisible Measure:* Let  $M$  be an infinitely divisible, independently scattered random measure on

$$\mathbb{P}^+ = \mathbb{R} \times \mathbb{R}^+ = \{(t, r) : t \in \mathbb{R}, r > 0\} \quad (8)$$

with the generating infinitely divisible distribution  $G$  having a moment generating function

$$\int_{\mathbb{R}} e^{qx} G(dx) = e^{-\rho(q)} \quad (9)$$

and the control measure

$$dm(t, r) = dt \mu(dr) \quad (10)$$

on  $\mathbb{P}^+$ . Independent scatteredness means that random variables  $M(A_1), \dots, M(A_n)$ ,  $n \geq 1$ , are independent whenever Borel sets  $A_i \subset \mathbb{P}^+$  are mutually disjoint. Infinite divisibility with the generating function  $G$  and the control measure  $m$  means that the moment generating function of  $M(A)$  with Borel set  $A \subset \mathbb{P}^+$  is given by

$$Ee^{qM(A)} = e^{-\rho(q)m(A)}. \quad (11)$$

*Infinitely Divisible Noise:* By definition, an IDC noise is a family of processes  $Q_r(t)$ ,  $r > 0, t \in \mathbb{R}$ , defined by

$$Q_r(t) = \frac{e^{M(C_r(t))}}{Ee^{M(C_r(t))}} \quad (12)$$

where

$$C_r(t) = \{(t', r') : r \leq r' \leq 1, t - \frac{r'}{2} \leq t' < t + \frac{r'}{2}\} \quad (13)$$

is a cone in  $\mathbb{P}^+$ . For each  $r > 0$ , the process  $Q_r(t)$ ,  $t \in \mathbb{R}$ , is stationary. We also let

$$Q_r^s(u) = \frac{Q_r(u)}{Q_s(u)} \quad (14)$$

which will be used in the proof of one of the main results, and set

$$\varphi(q) = \rho(q) - q\rho(1). \quad (15)$$

Note that  $\varphi(0) = \varphi(1) = 0$  and  $\varphi$  is concave. Moreover,  $\varphi(2) < 0$ . The trivial case  $\rho(q) = 0$  (yielding  $Q_r(t) \equiv 1$ ) is excluded throughout.

Examples of underlying distributions  $G$  are Gaussian, stable, gamma, compound Poisson laws. Several useful formulas regarding IDC noise are

$$EQ_r(t)^q = e^{-\varphi(q)m_r(0)} \quad (16)$$

$$EQ_r(t)Q_r(s) = e^{-\varphi(2)m_r(|t-s|)} \quad (17)$$

where

$$m_r(u) = m(C_r(0) \cap C_r(u)), \quad u \geq 0, r > 0. \quad (18)$$

(Relation (17) implicitly assumes that  $\varphi(2)$  exists. This may not be the case for some underlying distributions  $G$  such as most of stable distributions.) More generally, as in Lemma 1 of Bacry and Muzy [6, p. 457], one has, for  $0 \leq z_1 \leq z_2 \leq \dots \leq z_n$

$$EQ_r(z_1)Q_r(z_2) \dots Q_r(z_n) = \exp \left\{ - \sum_{1 \leq i < j \leq n} \alpha_{j-i+1} m_r(z_j - z_i) \right\} \quad (19)$$

where

$$\alpha_k = \varphi(k) - 2\varphi(k-1) + \varphi(k-2), \quad k \geq 2. \quad (20)$$

Since  $\varphi$  is a concave function, one has  $\alpha_k \leq 0$ .

*Scale Invariance and Infinitely Divisible Noise:* Throughout this work, we consider only those infinitely divisible noises whose control measure  $m$  is given by either

$$dm(t, r) = \begin{cases} dt \frac{cdt}{r^2}, & \text{if } 0 < r \leq 1 \\ 0, & \text{if } r > 1 \end{cases} \quad (21)$$

or

$$dm(t, r) = \begin{cases} dt \left( \frac{cdt}{r^2} + c\delta_{\{1\}}(dr) \right), & \text{if } 0 < r \leq 1 \\ 0, & \text{if } r > 1 \end{cases} \quad (22)$$

where  $c > 0$  is a constant and  $\delta_{\{1\}}(dr)$  denotes a point mass at  $r = 1$ . With this choice, we follow Muzy and Bacry [5], [6], Chainais *et al.* [9], [10]. The measures (21)–(22) are such that  $m_r(0)$  behaves as  $-c \ln r$  when  $r \rightarrow 0$  (cf. (102)–(103)), and are expected to lead to processes with truly multifractal properties.

The cases (21)–(22) are known as those of *scale invariance*. The case (22), considered in Bacry and Muzy [5], [6], is also referred to as that of *exact scale invariance*.

In the case of exact scale invariance, in particular, for  $t \in (0, 1)$

$$\{Q_{rt}(tu)\}_{u \in \mathbb{R}} \stackrel{d}{=} e^{\Omega_t} \{Q_r(u)\}_{u \in \mathbb{R}}, \quad (23)$$

where  $\stackrel{d}{=}$  stands for the equality of finite-dimensional distributions and  $\Omega_t$  is a random variable which is independent of  $\{Q_r(u)\}_{u \in \mathbb{R}}$  and has the moment generating function

$$Ee^{q\Omega_t} = t^{c\varphi(q)} \quad (24)$$

where  $\varphi(q)$  is given by (15). In the scale invariant case (21), using the notation (14), for  $0 < r < b < 1$

$$\{Q_r^b(bu)\}_{u \in \mathbb{R}} \stackrel{d}{=} \{Q_{r/b}(u)\}_{u \in \mathbb{R}}. \quad (25)$$

Again, only the cases (21)–(22) are considered in this work, and we will generally make no further reference to this, unless the focus is specifically on (21) or (22).

*Multifractal Infinitely Divisible Motion:* A truly multifractal process, called IDC motion in Chainais *et al.* [9, p. 1068], is obtained by integrating  $Q_r(t)$  over  $t$  and letting  $r \rightarrow 0$ , namely

$$X(t) = \lim_{r \rightarrow 0} \int_0^t Q_r(u) du. \quad (26)$$

We shall substitute  $Q_r$  into the integral (2) for  $Q$  and also let  $r \rightarrow 0$ . Note that the limit is outside the integral sign in (26). In fact, the limiting object  $\lim_{r \rightarrow 0} Q_r(u)$  is trivial: it converges to 0 almost surely and for almost all  $u$  (see Chainais *et al.* [9, Sec. III.A]). The limiting integral (26), on the other hand, is nondegenerate under suitable (and mild) assumptions. In case of scale invariant cascades, the corresponding assumption is

$$\exists \epsilon > 0 : \zeta_{\text{IDC}}(1 + \epsilon) > 1 \quad (27)$$

where

$$\zeta_{\text{IDC}}(q) = q + c\varphi(q) \quad (28)$$

or since  $\zeta_{\text{IDC}}(1) = 1$  and assuming  $\zeta_{\text{IDC}}$  is smooth

$$\zeta'_{\text{IDC}}(1) = 1 + c\varphi'(1) > 0 \quad (29)$$

(see Barral and Mandelbrot [4], Bacry and Muzy [6]).

### B. Fractional Wiener Integrals

We recall here several known facts on fractional Wiener integrals  $\int_0^t f(u) dB^\kappa(u)$  that will be used in the paper, where  $B^\kappa$ ,  $\kappa \in (-1/2, 1/2)$ , is fBm,  $f$  is a deterministic function, and  $t \in \mathbb{R}$  is fixed. Most of the facts below are taken from a review article by Pipiras and Taqqu [22]. We also exclude below the case  $\kappa = 0$  for which fractional Wiener integral is well known to be defined when  $f \in L^2[0, t]$ .

When  $\kappa \in (0, 1/2)$ , several classes of integrands  $f$  are available for fractional Wiener integrals. One such class is defined as

$$\mathcal{L}^\kappa = \left\{ f : \int_0^t \int_0^t |f(u)||f(v)||u-v|^{2\kappa-1} dudv < \infty \right\}. \quad (30)$$

It can be shown to be an inner product space with the inner product

$$\langle f, g \rangle_{\mathcal{L}^\kappa} = C_\kappa \int_0^t \int_0^t f(u)g(v)|u-v|^{2\kappa-1} du dv \quad (31)$$

where  $C_\kappa > 0$  is a constant. (The inner product space is known to be incomplete.)

When  $\kappa \in (-1/2, 0)$ , the commonly used class of integrands  $f$  is defined as

$$\mathcal{L}^\kappa = \{f : \exists \phi_f \in L^2[0, t] : f(u) = u^{-\kappa} (I_{t-}^{-\kappa} s^\kappa \phi_f(s))(u)\}. \quad (32)$$

It can be shown to be a Hilbert space with the inner product

$$\begin{aligned} \langle f, g \rangle_{\mathcal{L}^\kappa} &= C_\kappa \int_0^t s^{-2\kappa} (I_{t-}^\kappa u^\kappa f(u))(s) (I_{t-}^\kappa u^\kappa g(u))(s) ds \\ &= C_\kappa \int_0^t \phi_f(s) \phi_g(s) ds \end{aligned} \quad (33)$$

where, for  $\alpha > 0$

$$(I_{t-}^\alpha g)(s) = \frac{1}{\Gamma(\alpha)} \int_0^t g(u) (u-s)_+^{\alpha-1} du, \quad s \in [0, t] \quad (34)$$

is a fractional integral of order  $\alpha$  and, for  $\alpha \in (-1, 0)$

$$(I_{t-}^\alpha g)(u) = -\frac{1}{\Gamma(\alpha+1)} \frac{d}{du} \int_0^t g(s) (s-u)_+^\alpha ds, \quad u \in [0, t] \quad (35)$$

is a fractional derivative of order  $(-\alpha)$  (or fractional integral of negative order  $\alpha$ ).

For  $f \in \mathcal{L}^\kappa$ , a fractional Wiener integral, denoted as

$$\mathcal{I}^\kappa(f) = \int_0^t f(u) dB^\kappa(u), \quad t \in \mathbb{R} \quad (36)$$

is defined in the usual way as the  $L^2(\Omega)$ -limit of integrals  $\mathcal{I}^\kappa(f_n)$  for elementary functions  $f_n$  such that  $\langle f - f_n, f - f_n \rangle_{\mathcal{L}^\kappa} \rightarrow 0$ . (An elementary function  $g$  has a form  $g(u) = \sum_{k=1}^n g_k 1_{[a_k, b_k)}(u)$  and its fractional Wiener integral is defined as  $\mathcal{I}^\kappa(g) = \sum_{k=1}^n g_k (B^\kappa(b_k) - B^\kappa(a_k))$ .) By definition,  $\mathcal{I}^\kappa(f)$ ,  $f \in \mathcal{L}^\kappa$ , are Gaussian random variables with the covariance structure

$$E\mathcal{I}^\kappa(f)\mathcal{I}^\kappa(g) = \langle f, g \rangle_{\mathcal{L}^\kappa}. \quad (37)$$

Fractional derivatives and hence the case  $\kappa \in (-1/2, 0)$  are generally more difficult to deal with. The following formula, however, often facilitates computations even in the case  $\kappa \in (-1/2, 0)$ . For “nice” functions  $g$  and  $\alpha \in (-1, 0)$

$$\begin{aligned} (I_{t-}^\alpha g)(u) &= \frac{g(u)(t-u)^\alpha}{\Gamma(\alpha+1)} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (g(s) - g(u))(s-u)_+^{\alpha-1} ds. \end{aligned} \quad (38)$$

“Nice” means here, for example, step or  $C^1[0, t]$  functions. Observe that, unlike (35), the relation (38) does not involve derivatives. Another instance where the relation (38) holds is the following.

*Lemma 2.1:* Let  $\alpha \in (-1/2, 0)$  and  $g \in L^2[0, t]$  be such that  $g_{\parallel}^{(\alpha)} \in L^2[0, t]$ , where

$$\begin{aligned} g_{\parallel}^{(\alpha)}(u) &= \frac{|g(u)|(t-u)^\alpha}{\Gamma(\alpha+1)} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t |g(s) - g(u)|(s-u)_+^{\alpha-1} ds \end{aligned} \quad (39)$$

where  $(v)_+ = \max(0, v)$ . Then,  $I_{t-}^\alpha g$  is well defined, given by (38), and belongs to  $L^2[0, t]$ .

*Proof:* For  $\varepsilon \in [0, t]$ , consider

$$\begin{aligned} g_\varepsilon^{(\alpha)}(u) &= g(u) \frac{(t+\varepsilon-u)^\alpha}{\Gamma(\alpha+1)} \\ &+ \frac{1}{\Gamma(\alpha)} \int_u^t (g(s) - g(u))(s-u+\varepsilon)^{\alpha-1} ds. \end{aligned} \quad (40)$$

Then  $g_\varepsilon^{(\alpha)}$  belongs to  $L^2[0, t]$  and converges to  $g_0^{(\alpha)}$  in  $L^2[0, t]$ . Moreover

$$g_\varepsilon^{(\alpha)}(u) = \frac{\varepsilon^\alpha g(u)}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_u^t g(s) (s+\varepsilon-u)^{\alpha-1} ds.$$

These observations and the Fubini’s theorem (for the second equality below) allow us to conclude that

$$\begin{aligned} \int_v^t g_0^{(\alpha)}(u) du &= \lim_{\varepsilon \rightarrow 0} \int_v^t g_\varepsilon^{(\alpha)}(u) du \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\Gamma(\alpha+1)} \int_v^t g(s) (s-v+\varepsilon)^\alpha ds \\ &= \frac{1}{\Gamma(\alpha+1)} \int_v^t g(s) (s-v)^\alpha ds. \end{aligned}$$

This implies that  $I_{t-}^\alpha g$  is well defined and equals  $g_0^{(\alpha)}$ .  $\square$

*Corollary 2.1:* For  $\kappa \in (-1/2, 0)$ , if  $g \in L^2[0, t]$  and if  $g_{\parallel}^{[\kappa]} \in L^2[0, t]$ , where

$$\begin{aligned} g_{\parallel}^{[\kappa]}(u) &= \frac{|g(u)|(t-u)^\kappa}{\Gamma(\kappa+1)} \\ &+ \frac{u^{-\kappa}}{\Gamma(\kappa)} \int_0^t |s^\kappa g(s) - u^\kappa g(u)| (u-s)_+^{\kappa-1} ds \end{aligned} \quad (41)$$

then  $g$  belongs to  $\mathcal{L}^\kappa$  and

$$\begin{aligned} \phi_g(u) &= \frac{g(u)(t-u)^\kappa}{\Gamma(\kappa+1)} \\ &+ \frac{u^{-\kappa}}{\Gamma(\kappa)} \int_0^t (s^\kappa g(s) - u^\kappa g(u)) (u-s)_+^{\kappa-1} ds. \end{aligned} \quad (42)$$

### C. Fractional Wiener Integrals of IDC Noise

In Section III, we shall consider MRWs as the limits of fractional Wiener integrals

$$Z_r^\kappa(t) = \int_0^t Q_r(u) dB^\kappa(u) \quad (43)$$

as  $r \rightarrow 0$ , where  $Q_r(u)$  is an IDC noise defined in Section II-A and  $B^\kappa$  is fBm. Using the preliminaries of Sections II-A and

II-B, we state here that (43) is well defined for each fixed  $r > 0$ . See Section V for a proof.

*Proposition 2.1:* For  $\kappa \in (-1/2, 1/2)$ ,  $Q_r \in \mathcal{L}^\kappa$  a.s. for each fixed  $r > 0$  and, hence, fractional Wiener integral (43) is well defined a.s. for each  $r > 0$ .

*D. Important Notation and Cases*

Throughout this work, we consider only the case of scale invariance (21) or (22). It is convenient to introduce the following notation and cases. Let

$$\kappa_c = -c\varphi(2)/2 > 0. \tag{44}$$

Consider the cases

$$(A) : c\varphi(2) + 2\kappa > 0 \Leftrightarrow \kappa_c < \kappa < 1/2 \tag{45}$$

$$(E) : c\varphi(2) + 2\kappa = 0 \Leftrightarrow \kappa = \kappa_c \tag{46}$$

$$(B) : c\varphi(2) + 2\kappa < 0, \kappa \geq 0 \Leftrightarrow 0 \leq \kappa < \kappa_c \tag{47}$$

$$(N) : c\varphi(2) + 2\kappa < 0, \kappa < 0 \Leftrightarrow -1/2 < \kappa < 0 \tag{48}$$

where the notation (A), (E), and (B) stand for ‘‘Above,’’ ‘‘Equal to,’’ and ‘‘Below’’ the critical value  $\kappa_c$ , respectively, while (N) indicates ‘‘Negative’’ for  $\kappa < 0$ . Let also

$$\begin{aligned} \zeta(q) &= \begin{cases} (\kappa + \frac{1}{2})q + c\varphi(q), & \text{when (A)} \\ (\kappa_c + \frac{1}{2})q + c\varphi(q), & \text{when (E or B or N)} \end{cases} \\ &= (\kappa^* + \frac{1}{2})q + c\varphi(q) \end{aligned} \tag{49}$$

where

$$\kappa^* = \max\{\kappa, \kappa_c\}. \tag{50}$$

The function  $\zeta(q)$  will play the role of scaling exponents. The cases (E) and (B) will be refined below.

III. MRW: DEFINITION, CONVERGENCE, AND NUMERICAL SYNTHESIS

We wish to define Multifractal Random Walk (MRW) as the limit of fractional integrals  $Z_r^\kappa(t)$  in (43), as  $r \rightarrow 0$ . The following section, Section III-A, contains some preliminary technical results. Various types of convergence are considered in Section III-B, and numerical synthesis is discussed in Section III-C. Most of stated results of this section are proved in Section V.

*A. Calculations of Even-Order Moments*

The following technical result concerns even-order moments of  $Z_r^\kappa(t)$  when  $\kappa \in (0, 1/2)$ .

*Proposition 3.1:* Suppose  $p \in \mathbb{N}$  is such that

$$\zeta(2p) > 1 \tag{51}$$

where  $\zeta$  is defined in (49). Then, for fixed  $t \in \mathbb{R}$ , as  $r \rightarrow 0$

$$E|Z_r^\kappa(t)|^{2p} \sim C(p, t)n(r)^{2p} \tag{52}$$

where

$$n(r)^2 = \begin{cases} 1, & \text{when (A)} \\ \ln r^{-1}, & \text{when (E)} \\ r^{c\varphi(2)+2\kappa}, & \text{when (B)} \end{cases} \tag{53}$$

and  $C(p, t)$  is a nonzero constant which depends on  $p, t$  and also on  $\kappa, c$ , and  $\varphi$ .

Note that  $n(r)$  in (53) diverges as  $r \rightarrow 0$  in cases (E) and (B). The case  $\kappa \in (-1/2, 0)$  is much more cumbersome, and only the case of  $p = 1$  (second moment) is considered.

*Proposition 3.2:* In case (N), for fixed  $t \in \mathbb{R}$ , as  $r \rightarrow 0$

$$E|Z_r^\kappa(t)|^2 \sim C(t)n(r)^2 := C(t)r^{c\varphi(2)+2\kappa} \tag{54}$$

where  $C(t)$  is a nonzero constant which depends on  $t$  and also on  $\kappa, c$  and  $\varphi(2)$ .

The relations (52) and (54) above suggest that  $Z_r^\kappa$  may need to be normalized for convergence when  $0 \leq \kappa \leq \kappa_c$  or  $\kappa < 0$ . Hence, let also

$$\tilde{Z}_r^\kappa(t) = \frac{Z_r^\kappa(t)}{n(r)} \tag{55}$$

where  $n(r)$  is defined in (53) and (54).

*B. Various Types of Convergence*

We discuss here several types of convergence of the process  $\tilde{Z}_r^\kappa(t)$  defined in (55).

*1) Convergence of Moments:* Possibly the weakest type of convergence is that of (absolute) moments. Let  $\mathcal{M}$  be a subset of  $\mathbb{R}$ . We say that the  $\mathcal{M}$  moments of  $X_r$  converge to those of  $X$  (the  $\mathcal{M}$  moments of  $X_r$  converge, respectively), if  $E|X_r|^m \rightarrow E|X|^m$  for  $m \in \mathcal{M}$  ( $E|X_r|^m$  converge for  $m \in \mathcal{M}$ , respectively). Though the moment convergence is weak, it is quite appropriate for multifractals where their scaling properties are defined in terms of absolute moments.

Proposition 3.1 can now be restated as follows, and requires no separate proof.

*Theorem 3.1:* In cases (A), (E), and (B), the  $\mathcal{M}$  moments of  $\tilde{Z}_r^\kappa(t)$  converge, where

$$\mathcal{M} = \{2p : p \in \mathbb{N}, \zeta(2p) > 1\} \tag{56}$$

and  $\zeta$  is defined by (49).

*Remark 3.1:* The convergence of moments in Theorem 3.1 can be extended to that of linear combinations of finite-dimensional distributions of  $\tilde{Z}_r^\kappa$ , that is,  $a_1\tilde{Z}_r^\kappa(t_1) + \dots + a_n\tilde{Z}_r^\kappa(t_n)$ ,  $a_i, t_i \in \mathbb{R}, i = 1, \dots, n$ .

In case (N), by Proposition 3.2, the second moment of  $\tilde{Z}_r^\kappa(t)$  converges by construction. We conjecture that all  $\mathcal{M}$  moments specified in Theorem 3.1 also converge.

*2) Convergence in  $L^2$ :* Another useful type of convergence is that in  $L^2(\Omega)$ . In this regard, the following result is useful.

*Lemma 3.1:* For  $r \geq r^* > 0$ , we have  $EZ_r^\kappa(t)Z_r^{\kappa^*}(t) = EZ_r^\kappa(t)^2$ , where  $Z_r^{\kappa^*}(t)$  is defined by (43).

Propositions 3.1 and 3.2, and Lemma 3.1 above lead to the following result.

*Theorem 3.2:* For fixed  $t \in \mathbb{R}$ ,  $\tilde{Z}_r^\kappa(t)$  converges in  $L^2(\Omega)$  in case (A), and does not converge in  $L^2(\Omega)$  in cases (E), (B), and (N).

*Proof:* In case (A), we have  $\tilde{Z}_r^\kappa(t) = Z_r^\kappa(t)$ . Observe by Lemma 3.1 that, for  $r \geq r^* > 0$

$$E(Z_r^\kappa(t) - Z_{r^*}^\kappa(t))^2 = EZ_{r^*}^\kappa(t)^2 - EZ_r^\kappa(t)^2. \quad (57)$$

Hence, there is convergence in  $L^2(\Omega)$  if and only if  $EZ_{r^*}^\kappa(t)^2$  converges. This happens under (A) by Proposition 3.1.

For the last statement of the theorem, consider, for example, the case (E) with  $n(r) = \sqrt{\ln r^{-1}}$ . Then, by Lemma 3.1, for  $r \geq r^*$

$$\begin{aligned} E(\tilde{Z}_r^\kappa(t) - \tilde{Z}_{r^*}^\kappa(t))^2 &= E\left(\frac{Z_r^\kappa(t)}{\sqrt{\ln r^{-1}}} - \frac{Z_{r^*}^\kappa(t)}{\sqrt{\ln r^{*-1}}}\right)^2 \\ &= \frac{EZ_r^\kappa(t)^2}{\ln r} \left(2\sqrt{\frac{\ln r}{\ln r^*}} - 1\right) - \frac{EZ_{r^*}^\kappa(t)^2}{\ln r^*}. \end{aligned} \quad (58)$$

By Proposition 3.1,  $EZ_r^\kappa(t)^2/\ln r$ ,  $EZ_{r^*}^\kappa(t)^2/\ln r^*$  converge to the same nonzero constant as  $r, r^* \rightarrow 0$ . But  $r^*, r$  can be taken such that  $\lim_{r \rightarrow 0} \frac{\ln r}{\ln r^*} = a < 1$ , showing that  $\tilde{Z}_r^\kappa(t)$  does not converge in  $L^2(\Omega)$ .  $\square$

*Remark 3.2:* In case (A) when there is convergence in  $L^2(\Omega)$ , one may be inclined to write and to define the limit MRW as  $\int_0^t Q_0(u)dB^\kappa(u)$ . This is impossible because  $Q_0(u)$  is not defined as a regular process. (See also the discussion following (26).) On the other hand, one could still expect that the limit  $\lim_r \int_0^t Q_r(u)dB^\kappa(u)$  is characterized by the IDC motion  $X$  defined in (26). In fact, Ludeña [21] defines the limit as being conditionally Gaussian with the covariance structure

$$\int_0^t \int_0^s |u - v|^{2\kappa - 1} X(du)X(dv). \quad (59)$$

The appearance of this covariance structure suggests that the above integral is well defined (a.s.) for IDC motion  $X$ . Since  $|u - v|^{2\kappa - 1}$  blows up at  $u = v$ , this obviously imposes smoothness conditions on the motion  $X$ . A closer look at the argument of Ludeña [21] shows that (59) is understood as the limit of  $\int_0^t \int_0^s Q_r(u)Q_r(v)|u - v|^{2\kappa - 1} dudv$  as  $r \rightarrow 0$ . Whether it can be interpreted through  $X$  itself is an open question.

*Remark 3.3:* Observe that convergence in  $L^2(\Omega)$  for MRWs (case (A)) implies the condition (29) for nondegeneracy of IDC motion. Indeed, in case (A), one has  $\zeta(1) = \kappa + 1/2 < 1$ ,  $\zeta(2) = 2(\kappa - \kappa_c) + 1 > 1$ , and concave  $\zeta(q)$ . Assuming smoothness of  $\zeta$ , the latter conditions can happen only when  $\zeta'(1) > 0$ . But  $\zeta'(1) = \kappa + 1/2 + c\varphi'(1) < 1 + c\varphi'(1) = \zeta'_{\text{IDC}}(1)$  and hence  $\zeta'_{\text{IDC}}(1) > 0$ , that is, (29) holds. We do not believe this argument could be reversed in general.

*Remark 3.4:*  $L^2(\Omega)$ -convergence and nonconvergence in Theorem 3.2 can also be viewed from the following interesting

angle. For example, when  $\kappa = 0$ , if  $\int_0^t r^{-c\varphi(2)/2} Q_r(u)dB^0(u)$  converged, one would have that

$$\begin{aligned} E\left(\int_0^t \frac{Q_{r_1}(u)}{r_1^{c\varphi(2)/2}} dB^0(u) - \int_0^t \frac{Q_{r_2}(u)}{r_2^{c\varphi(2)/2}} dB^0(u)\right)^2 \\ = E\int_0^t \left(r_1^{-c\varphi(2)/2} Q_{r_1}(u) - r_2^{-c\varphi(2)/2} Q_{r_2}(u)\right)^2 du \rightarrow 0 \end{aligned}$$

as  $r_1, r_2 \rightarrow 0$ . Completeness of  $L^2(\Omega \times [0, t])$  would imply that  $r^{-c\varphi(2)/2} Q_r(u)$  converges to a regular process in  $L^2(\Omega \times [0, t])$ . But this would not be possible since  $r^{-c\varphi(2)/2} Q_r(u)$  is expected highly irregular as  $r \rightarrow 0$ . The same argument could also be applied when  $\kappa < 0$  for which the integrand space of fBm is complete but not for  $\kappa > 0$  when there is no completeness (see Section II-B). These arguments suggest that nonconvergence in Theorem 3.2 is, in fact, expected when  $\kappa \leq 0$ , and that convergence in  $L^2(\Omega)$  can only happen when  $\kappa > 0$ .

3) *Convergence in Distribution:* Another type of convergence in a weaker sense is that in distribution. In fact, convergence of  $\tilde{Z}_r^\kappa(t)$  in distribution takes place when  $\kappa = 0$ . (When  $\kappa = 0$ , there is no convergence in  $L^2(\Omega)$ .) To see this in the scale invariant case (21), observe that

$$\begin{aligned} \tilde{Z}_r^0(t) &= \frac{Z_r^0(t)}{r^{c\varphi(2)/2}} = r^{-c\varphi(2)/2} \int_0^t Q_r(u)dB(u) \\ &= r^{-c\varphi(2)/2} \int_0^t \frac{e^{M(C_r(u))}}{Ee^{M(C_r(u))}} dB(u) \\ &= r^{-c\varphi(2)/2} \int_0^t e^{M(C_r(u))} dB(u) \\ &= \int_0^t \left(\frac{e^{2M(C_r(u))}}{Ee^{2M(C_r(u))}}\right)^{1/2} dB(u) =: \int_0^t \tilde{Q}_r(u)^{1/2} dB(u) \end{aligned} \quad (60)$$

where we used the fact that  $Ee^{qM(C_r(u))} = r^{c\rho(q)}$ . Here,  $\tilde{Q}_r(u)$  is another IDC noise characterized by  $\tilde{\rho}(q) = \rho(2q)$  and  $\tilde{\varphi}(q) = \varphi(2q) - q\varphi(2)$ . The convergence in distribution of (60) is shown in Bacry and Muzy [6, p. 473]. The convergence takes place also in distribution in the space of functions.

We conjecture that the convergence in distribution also takes place in cases (E), (B), and (N). This is supported by the convergence of moments discussed in Section III-B.1, and by the following numerical simulations. See Section III-C below for more details on numerical synthesis.

Fig. 1 shows histograms of  $\tilde{Z}_r^\kappa(1)$  for  $r$  ranging from 1/2 to 1/1024, respectively, in the case  $\kappa < 0 < \kappa_c$  (case (N)),  $0 < \kappa < \kappa_c$  (case (B)), and  $0 < \kappa_c < \kappa$  (case (A)). Each histogram results from 40 000 realizations of  $\tilde{Z}_r^\kappa(1)$  for underlying compound Poisson cascades  $Q_r(t)$  such that  $\varphi(2) = -0.24$  and  $c = 1$ . Therefore, the critical situation corresponds to the choice  $\kappa_c = -c\varphi(2)/2 = 0.12$ . In all cases, the histograms obtained for various resolutions  $r$  superimpose quite clearly, and support the conjecture of convergence in distribution.

4) *Finer Convergence Results in Case (A):* In case (A), finer convergence than that in  $L^2(\Omega)$  can be established.

*Theorem 3.3:* Consider the case (A). Then, as  $r \rightarrow 0$ , the sequence of processes  $\{\tilde{Z}_r^\kappa(t)\}_{t \in [0,1]}$  converges a.s. (and hence in distribution) in the space of continuous functions  $C([0, 1], \mathbb{R})$ .

5) *Convergence in  $L^q$  and Refinement of Earlier Cases:* Various types of convergence above can be complemented by that in  $L^q(\Omega)$ ,  $1 < q \leq 2$ . We suppose  $\kappa \in (0, 1/2)$  and consider only the case of exact scale invariance (22) for simplicity. For  $\kappa \in (0, 1/2)$ , let

$$\tilde{\zeta}(q) = (\kappa + \frac{1}{2})q + c\varphi(q). \tag{61}$$

Note that  $\zeta(q)$  in (49) and the function  $\tilde{\zeta}(q)$  coincide in case (A) and are different in cases (E) and (B).

*Theorem 3.4:* Consider the case of exact scale invariance (22) and  $\kappa \in (0, 1/2)$ . Suppose there is  $1 < q \leq 2$  such that

$$\tilde{\zeta}(q) > 1. \tag{62}$$

Then, for all  $t \in \mathbb{R}$

$$\sup_{r \in (0,1)} E|Z_r^\kappa(t)|^q < \infty. \tag{63}$$

In particular, for all  $t \in \mathbb{R}$ ,  $Z_r^\kappa(t)$  converges a.s. and in  $L^q(\Omega)$  as  $r \rightarrow 0$ .

Here is where Theorem 3.4 fits in with previous results. In case (A), by Theorem 3.2, there is convergence in  $L^2(\Omega)$  and hence that in  $L^q(\Omega)$  for all  $1 < q < 2$ . Convergence almost surely is also proved in Theorem 3.3. Hence, in this case, Theorem 3.4 does not yield any new results. On the other hand, new results emerge in cases (E) and (B). Two cases should now be distinguished.

Case 1:  $\tilde{\zeta}'(2) > 0$ ,

Case 2:  $\tilde{\zeta}'(2) < 0$ ,

where  $\tilde{\zeta}'(q)$  indicates the derivative of  $\tilde{\zeta}(q)$ , and  $\tilde{\zeta}(q)$  is defined in (61).

In cases (E) and (B),  $\tilde{\zeta}(q) = (\kappa + 1/2)q + c\varphi(q)$  is concave and such that  $\tilde{\zeta}(0) = 0$ ,  $\tilde{\zeta}(1) = \kappa + 1/2 < 1$ , and  $\tilde{\zeta}(2) = 1 + 2(\kappa - \kappa_c) \leq 1$ . Moreover,  $\zeta(q) = (\kappa_c + 1/2)q + c\varphi(q)$  is also concave and such that  $\zeta(0) = 0$ ,  $\zeta(1) = \kappa_c + 1/2$ ,  $\zeta(2) = 1$ , and  $\tilde{\zeta}(q) \leq \zeta(q)$ . In Case 1, note that  $\zeta'(2p) \geq \tilde{\zeta}'(2) > 0$ . Then, by concavity, one expects that  $\zeta(2p) > 1$  even for some  $p > 1$ , and that condition (62) does not hold for any  $1 < q \leq 2$ . This corresponds to the situation considered in Proposition 3.1 and Theorem 3.1.

On the other hand, in Case 2, by concavity, one expects that condition (62) might hold for some  $1 < q < 2$ . (In particular, in case (E), one expects this condition to hold for  $q$  sufficiently close to 2.) In this case, by Theorem 3.4, there is convergence almost surely and in  $L^q(\Omega)$  without any normalization. Note also that, if condition (62) holds for some  $1 < q < 2$ , one cannot expect that  $\zeta(2p) > 1$  for  $p > 1$ . Indeed, supposing  $\tilde{\zeta}(q) > 1$  for some  $1 < q < 2$  yields  $\zeta(q) > 1$ . Using concavity of  $\zeta(q)$  and  $\zeta(2) = 1$ , one expects that  $\zeta(2p) < 1$  for  $p > 1$ .

In order to distinguish between above Cases 1 and 2, we shall use the following notation. Cases (E) and (B) will refer to the

earlier cases (E) and (B) when  $\tilde{\zeta}'(2) > 0$ . When  $\tilde{\zeta}'(2) < 0$ , cases (E) and (B) will be denoted as

$$(E^*) \text{ and } (B^*)$$

respectively. In cases (E<sup>\*</sup>) and (B<sup>\*</sup>), we set  $\tilde{Z}_r^\kappa(t) = Z_r^\kappa(t)$ .

From a practical perspective, cases (E) and (B) appear to be more relevant than cases (E<sup>\*</sup>) and (B<sup>\*</sup>). First, most known examples of IDC noise, in fact, fall into cases (E) and (B). Second, in cases (E<sup>\*</sup>) and (B<sup>\*</sup>), one expects the limit  $\tilde{Z}_r^\kappa(t)$  of  $Z_r^\kappa(t)$  to have infinite second moment. For these reasons, in simulations and Section IV below, we shall focus on cases (E) and (B), rather than cases (E<sup>\*</sup>) and (B<sup>\*</sup>).

6) *Summary of Convergence:* In Table I, we summarize the convergence results for  $\tilde{Z}_r^\kappa$ . We will refer to the limit by the following name.

*Definition 3.1:* The limit process  $\tilde{Z}^\kappa$  of  $\tilde{Z}_r^\kappa$  will be called Multifractal Random Walk (MRW) as fractional Wiener integral or simply MRW, whenever it exists.

7) *Using Arbitrary Powers of IDC Noise:* It is interesting to compare the above definitions of MRWs to those when considering integrals

$$Z_r^{\kappa,\beta}(t) = \int_0^t (Q_r(u))^\beta dB^\kappa(u) \tag{64}$$

where the IDC noise  $Q_r(u)$  has a power  $\beta > 0$ . When  $\beta = 1/2$ , integrals (64) and their limits appear, for example, in Bacry and Muzy [6], Ludeña [21]. As already hinted in Section III-B.3, the case of integrals (64) can be reduced to that of integrals (43) considered above. We state this as an elementary result next.

*Proposition 3.3:* For  $\beta > 0$ , we have

$$\begin{aligned} Z_r^{\kappa,\beta}(t) &= \int_0^t (Q_r(u))^\beta dB^\kappa(u) \\ &= C_0 r^{c\varphi(\beta)} \int_0^t \tilde{Q}_r(u) dB^\kappa(u) =: C_0 r^{c\varphi(\beta)} \tilde{Z}_r^\kappa(t) \end{aligned} \tag{65}$$

where  $C_0 = e^{-c\varphi(\beta)}$  and  $C_0 = 1$  in the cases (21) and (22), respectively,  $\tilde{Q}_r$  is an IDC noise characterized by the same control measure  $m$ , and

$$\tilde{\varphi}(q) = \varphi(\beta q) - q\varphi(\beta). \tag{66}$$

*Proof:* Observe that

$$\begin{aligned} (Q_r(u))^\beta &= \frac{e^{\beta M(C_r(u))}}{(E e^{M(C_r(u))})^\beta} \\ &= \frac{E e^{\beta M(C_r(u))}}{(E e^{M(C_r(u))})^\beta} \frac{e^{\beta M(C_r(u))}}{E e^{\beta M(C_r(u))}} \\ &=: C_0 r^{c\varphi(\beta)} \frac{e^{\tilde{M}(C_r(u))}}{E e^{\tilde{M}(C_r(u))}} =: C_0 r^{c\varphi(\beta)} \tilde{Q}_r(u) \end{aligned}$$

where  $\tilde{M}(A) = \beta M(A)$  is an infinitely divisible measure characterized by

$$E e^{q\tilde{M}(A)} = E e^{\beta q M(A)} = e^{-\rho(\beta q)m(A)}$$

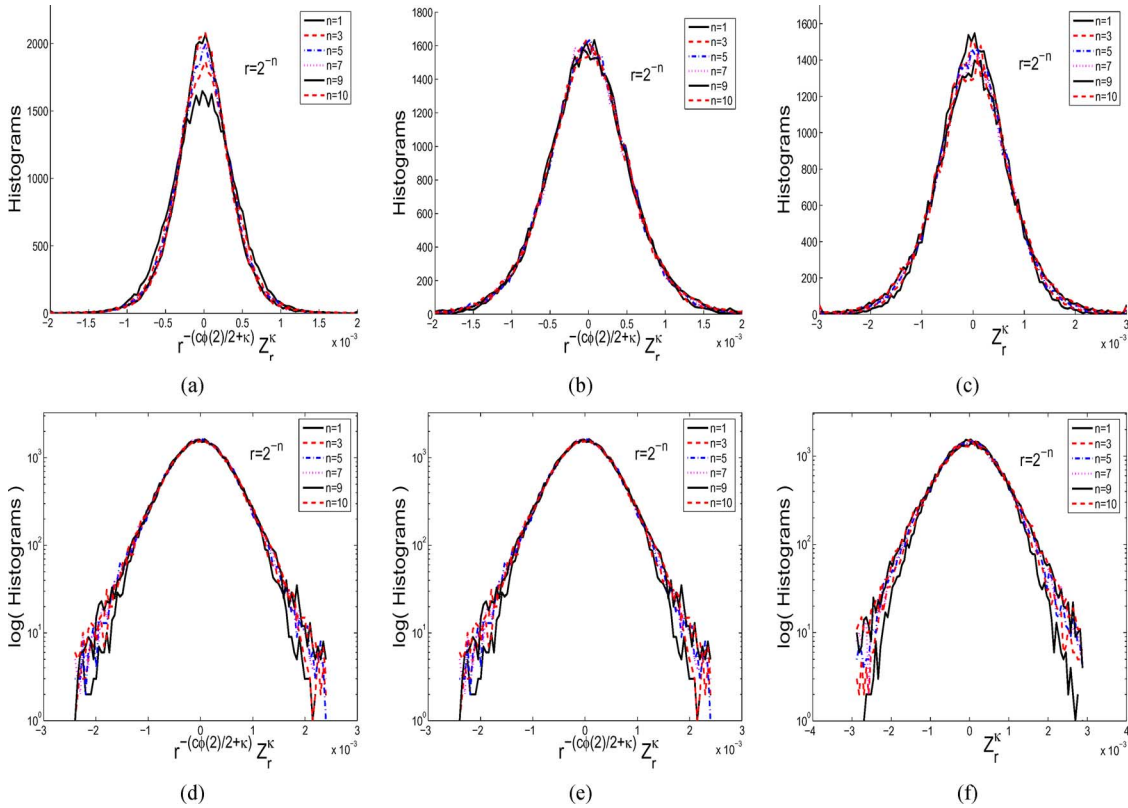


Fig. 1. Histograms of  $\tilde{Z}_r^\kappa(1)$  for various resolutions  $r$  ranging from  $1/1024$  to  $1/2$ . First row: histograms for (a)  $\kappa = -0.1 < 0 < \kappa_c$ , (b)  $0 < \kappa = 0.05 < \kappa_c$ , (c)  $\kappa_c < \kappa < 0.2$ . Second row: log-histograms for (d)  $\kappa = -0.1 < 0 < \kappa_c$ , (e)  $0 < \kappa = 0.05 < \kappa_c$ , (f)  $\kappa_c < \kappa < 0.2$ .

TABLE I  
CONVERGENCE OF  $\tilde{Z}_r^\kappa$  IN THE SCALE INVARIANT CASE

Case	Parameter range	$\tilde{Z}_r^\kappa$ converges in
(A)	$\kappa_c < \kappa < 1/2$	$L^2(\Omega)$ ; distribution; (suitable) moments; almost surely
(E), (B)	$0 \leq \kappa \leq \kappa_c, \tilde{\zeta}'(2) > 0$	(suitable) moments; not $L^2(\Omega)$ ; distribution when $\kappa = 0$ ; conjectured for distribution when $\kappa > 0$
(E*), (B*)	$0 \leq \kappa \leq \kappa_c, \tilde{\zeta}'(2) < 0$ $\tilde{\zeta}(q) > 1$ for some $q \in (1, 2)$	$L^q(\Omega)$ ; almost surely
(N)	$-1/2 < \kappa < 0$	not $L^2(\Omega)$ ; conjectured for distribution and moments

and hence  $\bar{\rho}(q) = \rho(\beta q)$  and  $\bar{\varphi}(q) = \bar{\rho}(q) - q\bar{\rho}(1) = \rho(\beta q) - q\rho(\beta) = \varphi(\beta q) - q\varphi(\beta)$ .  $\square$

As a consequence of Proposition 3.3 and Theorem 3.2, for example,

$$r^{-c\varphi(\beta)} Z_r^{\kappa, \beta}(t)$$

converges in  $L^2(\Omega)$  when

$$\kappa > -\frac{c\bar{\varphi}(2)}{2} = -\frac{c(\varphi(2\beta) - 2\varphi(\beta))}{2}.$$

In case  $\beta = 1/2$ , this happens when  $\kappa > c\varphi(1/2)$ . A discussion related to this section can also be found in Section IV-F below.

C. Numerical Synthesis

The process  $Z_r^\kappa$  in (43) is synthesized numerically as follows. To produce  $N$  samples  $Z_r^\kappa(n/N), n = 0, \dots, N - 1$ , at

equally spaced times on the interval  $[0, 1)$ , we generate  $K = RN$  equally spaced samples  $B^\kappa(k/K)$  and  $Q_r(k/K), k = 0, \dots, K - 1$ , of fBm  $B^\kappa$  and IDC noise  $Q_r$ . The integer constant  $R$  is referred to as the oversampling factor. Continuous sums in (43) are then approximated using Riemann sums

$$Z_r^\kappa\left(\frac{n}{N}\right) \simeq \sum_{k=1}^{nR} Q_r\left(\frac{k}{K}\right) \left( B^\kappa\left(\frac{k}{K}\right) - B^\kappa\left(\frac{k-1}{K}\right) \right). \quad (67)$$

The samples  $B^\kappa(k/K)$  of fBm are synthesized numerically using the so-called circulant matrix embedding method. This is not detailed here and the reader is referred, e.g., to Bardet *et al.* [23] for a complete introduction.

The synthesis of an IDC noise  $Q_r$  first requires the choices of theoretical quantities: an infinitely divisible measure  $M$  in (9) with its control measure (10); the triangle-shaped cone in (13).



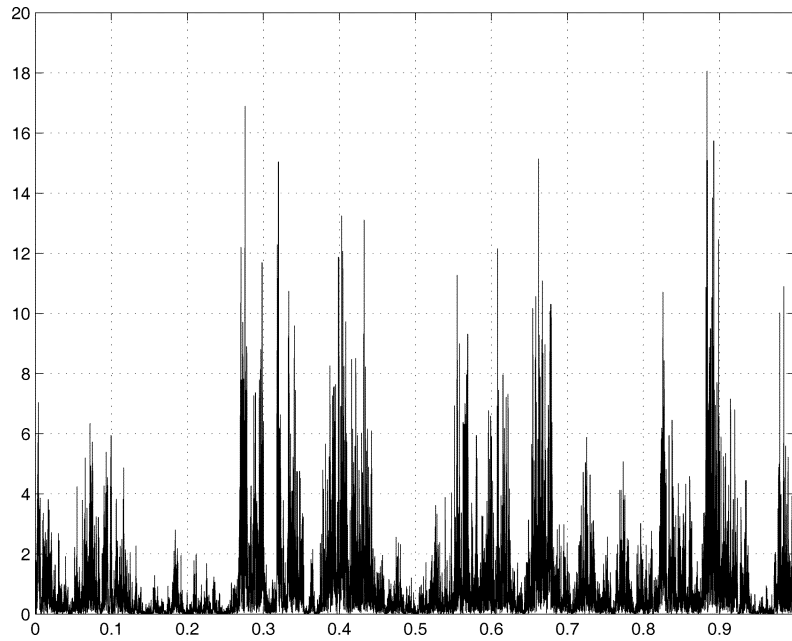


Fig. 2. A sample path of an IDC noise (CPC)  $Q_r$ .

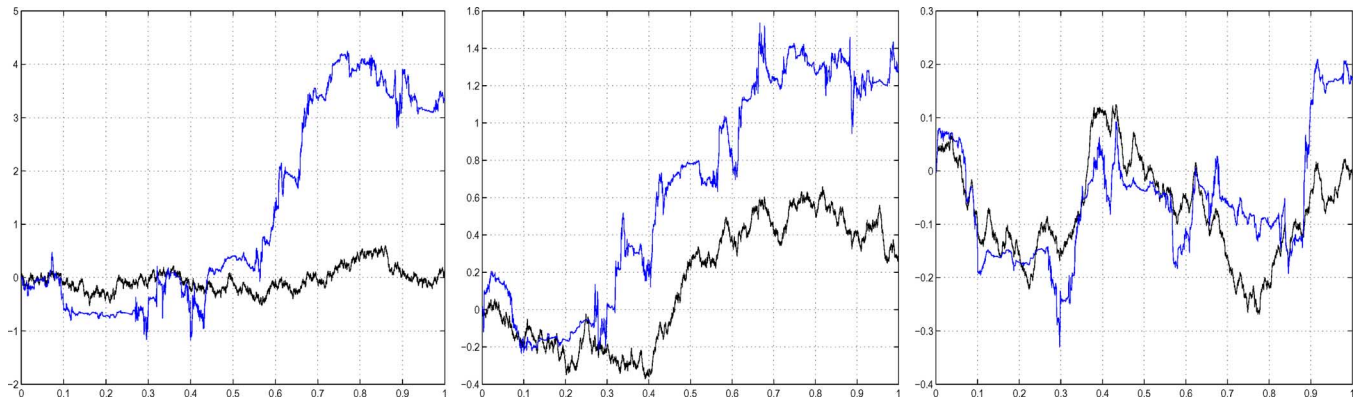


Fig. 3. Sample paths of MRW. Sample path of  $\tilde{Z}^\kappa$  (blue-gray) together with that of  $B^\kappa$  (black) from which it is obtained by combination with the sample path of  $Q_r$  shown in Fig. 2. From left to right:  $\kappa < 0 < \kappa_c$ ,  $0 < \kappa < \kappa_c$ ,  $0 < \kappa_c < \kappa$ .

In practice, the resolution parameter  $r$  in  $Q_r$  needs to be chosen such that  $r \geq 1/K$ . The samples  $Q_r(k/K)$  are generated using the algorithms thoroughly described in Chainais *et al.* [9] and not recalled here.

Examples of sample paths of  $Z_r^\kappa$  are produced using a specific type of IDC noise referred to as compound Poisson cascades (CPC). The same sample path of the CPC  $Q_r$ , shown in Fig. 2, is used to produce the sample paths of  $\tilde{Z}^\kappa$  for the three different cases,  $\kappa < 0 < \kappa_c$ ,  $0 < \kappa < \kappa_c$ , and  $0 < \kappa_c < \kappa$ , cf. Fig. 3. For illustration purposes, Fig. 4 compares the sample paths of MRW  $Z^\kappa$  against those of the so-called fBm in multifractal time

$$V^\kappa(t) = B^\kappa(X(t)) \tag{68}$$

where  $X(t)$  is the IDC motion in (26), obtained using the same sample paths of  $B^\kappa(t)$  and  $Q_r(t)$ . For definitions of fBm in multifractal time, the reader is referred to Mandelbrot [13], Riedi [1]. It is analyzed in Chainais *et al.* [9] and Bacry and Muzy [5], [6]. Note also that, thanks to Riemann sums (67), the numerical

synthesis of MRWs is much easier than that of fBm in multifractal time, which is based on resampling of fBm.

#### IV. PROPERTIES OF MRWS

We examine here several properties of MRWs. Since no type of convergence was established in case (N) in Section III-B, this case will be excluded from theoretical results below.

##### A. Stationary Increments and Long Range Dependence

In case (A), MRW  $\tilde{Z}^\kappa$  has strictly stationary increments because  $Q_r(u)$  is stationary and  $B^\kappa(u)$  has stationary increments. The time series

$$X_k = \tilde{Z}^\kappa((k+1)\tau) - \tilde{Z}^\kappa(k\tau), \quad k \in \mathbb{Z}$$

for some  $\tau > 0$ , of the increments of MRW  $\tilde{Z}^\kappa$  is therefore stationary. Similarly, in cases (E) and (B), MRW  $\tilde{Z}^\kappa$  has second-order stationary increments and the series  $X_k$  is second-order

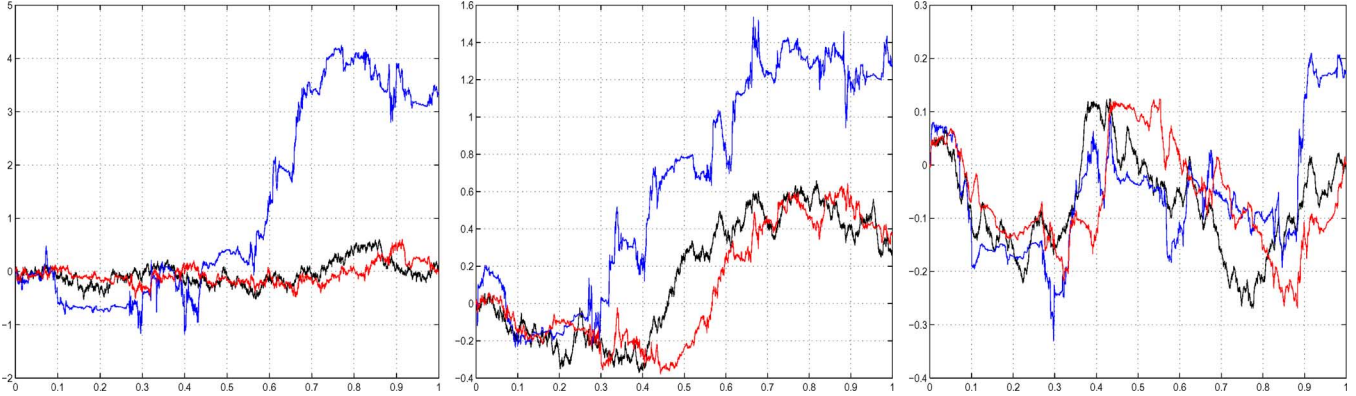


Fig. 4. Superimposition of MRW and fBm in multifractal time. Using the sample paths of  $Q_r$  shown in Fig. 2, superimpositions of the sample paths of  $B^\kappa$  (black),  $\tilde{Z}^\kappa$  (blue-gray),  $V^\kappa$  (red-light gray). From left to right:  $\kappa < 0 < \kappa_c$ ,  $0 < \kappa < \kappa_c$ ,  $0 < \kappa_c < \kappa$ .

stationary. The following result contains basic properties of  $\tilde{Z}^\kappa$  and  $X_k$ . See Section V for a proof.

*Proposition 4.1:*

(i) In case (A),  $\tilde{Z}^\kappa$  has zero mean and the covariance structure

$$E\tilde{Z}^\kappa(s)\tilde{Z}^\kappa(t) = C_\kappa \int_0^s \int_0^t EQ_0(u)Q_0(v) |u-v|^{2\kappa-1} dudv \quad (69)$$

where  $EQ_0(u)Q_0(v)$  is understood formally through (17), (102) and (103) by substituting  $r = 0$ . In particular, for  $k \geq 1/\tau + 1$ , one has

$$\begin{aligned} EX_kX_0 &= \frac{C_\kappa \tau^{2\kappa+1}}{2\kappa(2\kappa+1)} (|k+1|^{2\kappa+1} + |k-1|^{2\kappa+1} - 2|k|^{2\kappa+1}) \\ &\sim C_\kappa \tau^{2\kappa+1} |k|^{2\kappa-1} \end{aligned} \quad (70)$$

as  $k \rightarrow \infty$ .

(ii) In cases (E) and (B),  $\tilde{Z}^\kappa$  has the covariance

$$E\tilde{Z}^\kappa(s)\tilde{Z}^\kappa(t) = Cs, \quad \text{for } 0 \leq s \leq t. \quad (71)$$

In particular,  $EX_kX_0 = 0$ , for  $k \neq 0$ . For example, in case (22), the constant in (71) is given by

$$C = 2C_\kappa \left( -\frac{1}{c\varphi(2) + 2\kappa} + e^{-c\varphi(2)} \int_0^1 dw w^{2\kappa-1} e^{c\varphi(2)w} \right). \quad (72)$$

Time series  $X_k$  with the autocovariance function satisfying (70) with  $-1 < 2\kappa - 1 < 0$  as  $k \rightarrow \infty$  is known as long range dependent. Thus, according to Proposition 4.1,  $\tilde{Z}^\kappa$  is characterized only by two regimes at large scales: its increments are long range dependent for  $\kappa_c < \kappa < 1/2$  while it has uncorrelated increments for  $0 \leq \kappa \leq \kappa_c$ .

*B. Finiteness of Moments*

The following result characterizes finiteness of even-order moments of MRWs.

*Proposition 4.2:* Let  $p \in \mathbb{N}$ . In cases (A), (E), and (B), the  $(2p)$ th moment  $E|\tilde{Z}^\kappa(t)|^{2p}$  is finite when

$$\zeta(2p) > 1 \quad (73)$$

where  $\zeta(q)$  is defined in (49).

The function  $\zeta(q)$  plays the role of scaling exponents below. The condition (73) is therefore a natural and expected condition on finiteness of moments in the multifractal literature. In case(A), the condition (73) was conjectured by Ludeña [21].

*C. The Special Case of Exact Scale Invariance*

The case of exact scale invariance (22) is special in the sense that the corresponding MRW has the following scaling property.

*Proposition 4.3:* In the case of exact scale invariance, we have, for  $a \in (0, 1)$

$$\{\tilde{Z}^\kappa(at)\}_{t \in [0,1]} \stackrel{d}{=} a^{\kappa+\frac{1}{2}} e^{\Omega_a} \{\tilde{Z}^\kappa(t)\}_{t \in [0,1]} \quad (74)$$

in case (A) with  $\stackrel{d}{=}$  denoting the equality in the sense of finite-dimensional distributions, and

$$\{\tilde{Z}^\kappa(at)\}_{t \in [0,1]} \stackrel{m}{=} a^{\kappa_c+\frac{1}{2}} e^{\Omega_a} \{\tilde{Z}^\kappa(t)\}_{t \in [0,1]} \quad (75)$$

in cases (E) and (B), with  $\stackrel{m}{=}$  denoting the equality of even-order moments (from the set (56)) of linear combinations of finite-dimensional distributions. In (74) and (75),  $\Omega_a$  is a random variable independent of  $\{\tilde{Z}^\kappa(t)\}_{t \in [0,1]}$  and with the moment generating function (24).

*Proof:* In case (A), observe from (23) and  $(\kappa + \frac{1}{2})$ -self-similarity of fBm that

$$\begin{aligned} \tilde{Z}_{ar}^\kappa(at) &= \int_0^{at} Q_{ar}(u) dB^\kappa(u) \\ &\stackrel{d}{=} \int_0^t Q_{ar}(ax) dB^\kappa(ax) \\ &\stackrel{d}{=} a^{\kappa+\frac{1}{2}} e^{\Omega_a} \int_0^t Q_r(x) dB^\kappa(x). \end{aligned} \quad (76)$$

The result (74) follows by letting  $r \rightarrow 0$ .

In case (B), we have

$$\begin{aligned} & (ar)^{-(c\varphi(2)+2\kappa)/2} \int_0^{at} Q_{ar}(u) dB^\kappa(u) \\ & \stackrel{m}{=} a^{-(c\varphi(2)+2\kappa)/2} a^{\kappa+\frac{1}{2}} e^{\Omega_a r} a^{-(c\varphi(2)+2\kappa)/2} \int_0^t Q_r(x) dB^\kappa(x). \end{aligned} \tag{77}$$

The result (75) follows from Proposition 3.1 by letting  $r \rightarrow 0$ . The same relation can be deduced similarly in case (E).  $\square$

The relation (74) differs from the usual self-similarity in the presence of random variables  $e^{\Omega_a}$ . Proposition 4.3 above directly yields the scaling behavior of moments of MRWs (see Section IV-D.1).

D. Scaling Properties of Moments of MRWs

We study here the scaling properties of moments of MRW  $\tilde{Z}^\kappa$ . These concern the asymptotic behavior of the moments  $E|\tilde{Z}^\kappa(t)|^q$  as  $t \rightarrow 0$ .

1) *The Case of Exact Scale Invariance:* A particularly easy case is that of exact scale invariance (22). Taking  $a = t < 1$  and  $t = 1$  in (74), yields

$$\begin{aligned} E|\tilde{Z}^\kappa(t)|^q &= t^{(\kappa+\frac{1}{2})q} E e^{q\Omega_t} E|\tilde{Z}(1)|^q \\ &= t^{(\kappa+1/2)q+c\varphi(q)} E|\tilde{Z}(1)|^q \end{aligned}$$

in case (A). In cases (E) and (B), relation (75) yields

$$E|\tilde{Z}^\kappa(t)|^q = t^{(\kappa_c+1/2)q+c\varphi(q)} E|\tilde{Z}(1)|^q$$

for even-order moments  $q$ . Combining the two relations gives the following result.

*Theorem 4.1:* In the case of exact scale invariance (22) and (A), (E), and (B), we have, for  $t \in (0, 1)$

$$E|\tilde{Z}^\kappa(t)|^q = C_q t^{\zeta(q)} \tag{78}$$

where  $\zeta(q)$  is defined in (49), and  $q > 0$  is such that  $E|\tilde{Z}^\kappa(1)|^q < \infty$ .

The condition  $E|\tilde{Z}^\kappa(1)|^q < \infty$  holds for all even  $q = 2p$  in  $\mathcal{M}$  defined by (56). Note also that, in cases (E) and (B), scaling exponents  $\zeta(q)$  depend only on  $\kappa_c$ , not  $\kappa$ .

*Remark 4.1:* In case (A), note that (78) implies, in particular,  $E|\tilde{Z}^\kappa(t)|^2 = C_2 t^{2\kappa+1+c\varphi(2)}$ , as  $t \rightarrow 0$ . On the other hand, as  $t \rightarrow \infty$ , it follows from (69) that  $E|\tilde{Z}^\kappa(t)|^2 \sim \tilde{C}_2 t^{2\kappa+1}$ . The second moment thus scales differently at small and large scales. Note also that this is expected since small scales are influenced by  $Q_r$  while larger scales are dominated by fBm.

2) *The Case of Non-Exact Scale Invariance:* In the case of non-exact scale invariance (21), the moments of MRW do not scale exactly. However, the scaling (78) is expected as  $t \rightarrow 0$ . We provide here such a result adapting the approach in Theorem 1 of Chainais *et al.* [9, p. 1069], which concerns the scaling properties of IDC motion. (Another possibility is to follow the

approach used for Lemma 4 in Bacry and Muzy [6].) Recall the notation  $Q_r^s(u)$  defined in (14), and let  $q > 0$  be fixed. For  $0 < b < 1, t \in (0, 1]$  and  $k \in \mathbb{N}$ , let also

$$\begin{aligned} \tau_{b,q}^{(k)}(t) &= \\ & \frac{E \sup_{0 \leq u, v \leq b^k t} \left| Q_{b^k}^{b^{k-1}}(u)^{\frac{q}{2}} Q_{b^k}^{b^{k-1}}(v)^{\frac{q}{2}} - Q_{b^k}^{b^{k-1}}(0)^q \right|}{E Q_{b^k}^{b^{k-1}}(0)^q}. \end{aligned} \tag{79}$$

*Theorem 4.2:* Suppose  $\kappa \in (0, 1/2)$  and consider the case of scale invariance (21). Let  $q > 0$  be fixed. Suppose that, for  $t \in (0, 1]$

$$E|\tilde{Z}_r^\kappa(t)|^q \rightarrow E|\tilde{Z}^\kappa(t)|^q < \infty, \quad \text{as } r \rightarrow 0 \tag{80}$$

and that, for  $t \in (0, 1]$  and  $k \in \mathbb{N}$

$$\tau_{b,q}^{(k)}(t) \leq C_{b,q} t^\nu \tag{81}$$

with  $\nu > 0$ . Then, for  $t \in (0, 1]$  and some constants  $\underline{C}_q, \overline{C}_q$

$$\underline{C}_q t^{\zeta(q)} \leq E|\tilde{Z}^\kappa(t)|^q \leq \overline{C}_q t^{\zeta(q)} \tag{82}$$

where  $\zeta(q)$  is defined in (49).

See Section V for a proof of the theorem. In regard to the assumption (81), observe that, by the Hölder's inequality and the inequality  $|a - b|^2 \leq |a^2 - b^2|, a, b > 0$

$$\begin{aligned} & E \sup_{0 \leq u, v \leq b^k t} \left| Q_{b^k}^{b^{k-1}}(u)^{\frac{q}{2}} Q_{b^k}^{b^{k-1}}(v)^{\frac{q}{2}} - Q_{b^k}^{b^{k-1}}(0)^q \right| \\ & \leq E \sup_{0 \leq u \leq b^k t} Q_{b^k}^{b^{k-1}}(u)^{\frac{q}{2}} \sup_{0 \leq v \leq b^k t} \left| Q_{b^k}^{b^{k-1}}(v)^{\frac{q}{2}} - Q_{b^k}^{b^{k-1}}(0)^{\frac{q}{2}} \right| \\ & \quad + E Q_{b^k}^{b^{k-1}}(0)^{\frac{q}{2}} \sup_{0 \leq u \leq b^k t} \left| Q_{b^k}^{b^{k-1}}(u)^{\frac{q}{2}} - Q_{b^k}^{b^{k-1}}(0)^{\frac{q}{2}} \right| \\ & \leq \left\{ \left( E \sup_{0 \leq u \leq b^k t} Q_{b^k}^{b^{k-1}}(u)^q \right)^{1/2} + \left( E Q_{b^k}^{b^{k-1}}(0)^q \right)^{1/2} \right\} \\ & \quad \times \left\{ E \sup_{0 \leq u \leq b^k t} \left| Q_{b^k}^{b^{k-1}}(u)^q - Q_{b^k}^{b^{k-1}}(0)^q \right| \right\}^{1/2}. \end{aligned} \tag{83}$$

Hence

$$\begin{aligned} \tau_{b,q}^{(k)}(t) & \leq \left\{ \left( \frac{E \sup_{0 \leq u \leq b^k t} Q_{b^k}^{b^{k-1}}(u)^q}{E Q_{b^k}^{b^{k-1}}(0)^q} \right)^{1/2} + 1 \right\} \\ & \quad \times \left\{ \frac{E \sup_{0 \leq u \leq b^k t} \left| Q_{b^k}^{b^{k-1}}(u)^q - Q_{b^k}^{b^{k-1}}(0)^q \right|}{E Q_{b^k}^{b^{k-1}}(0)^q} \right\}^{1/2} \\ & =: \left\{ \left( \frac{E \sup_{0 \leq u \leq b^k t} Q_{b^k}^{b^{k-1}}(u)^q}{E Q_{b^k}^{b^{k-1}}(0)^q} \right)^{1/2} + 1 \right\} \tilde{\tau}_{b,q}^{(k)}(t) \\ & \leq \left( (\tilde{\tau}_{b,q}^{(k)}(t))^2 + 1 \right)^{1/2} + \tilde{\tau}_{b,q}^{(k)}(t). \end{aligned} \tag{84}$$

The functions  $\tilde{\tau}_{b,q}^{(k)}(t)$  are used in Chainais *et al.* [9] in the assumption similar to the assumption (81). Sufficient conditions for these functions to be bounded by the function  $C_{b,q} t^\nu, \nu > 0$ , are provided (see Corollary 2 in Chainais *et al.* [9]).

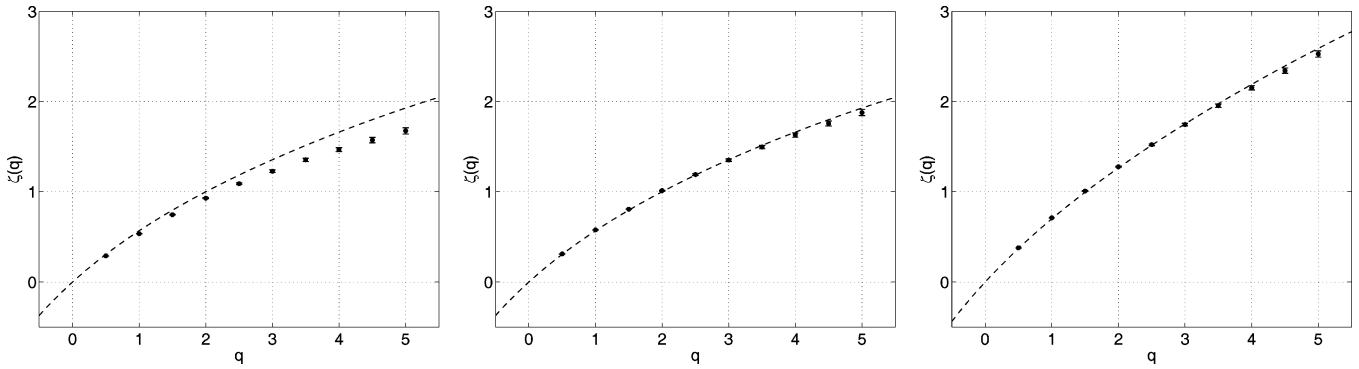


Fig. 5. Scaling exponents from increments. For  $\kappa = -0.1 < 0$ ,  $0 < \kappa = 0.02 < \kappa_c$ , and  $0 < \kappa_c < \kappa = 0.2$  (from left to right), dashed lines: theoretical scaling exponents; solid points: scaling exponents estimated using increment based structure functions; for  $q \in [0, q_*^+]$ . The agreement is very satisfactory when  $\kappa > 0$ . The case  $\kappa < 0$  is more difficult, requiring the use of higher oversampling rate  $R$ , and hence higher practical difficulties (computational cost and memory issues).

*Remark 4.2:* When  $0 \leq \kappa \leq \kappa_c$  (cases (E) and (B)), the scaling behavior of MRWs is the same according to Theorems 4.1 and 4.2. In particular, from a practical perspective, this suggests that the MRW with  $\kappa = 0 < \kappa_c$  could be used to capture the corresponding scaling behavior. The case  $\kappa = 0$  is obviously simpler than that of  $\kappa \neq 0$ .

*Remark 4.3:* As in Chainais *et al.* [9], Theorem 4.2 can be reformulated for control measures  $m$  that are more general than in (21). In this case, the term  $t^{\zeta(q)}$  in (82) gets replaced by  $t^{(\kappa_* + \frac{1}{2})q} e^{-\varphi(q)m(C_t(0))}$ , where  $\kappa_*$  is defined in (50).

### E. Numerical Illustrations

We now intend to illustrate practically the scaling behaviors reported in (78) and (82). These power law behaviors of  $\tilde{Z}^\kappa$  translate into those of its stationary increments as

$$E|\tilde{Z}^\kappa(t + a\tau_0) - \tilde{Z}^\kappa(t)|^q = C'_q |a|^{\zeta(q)} \quad (85)$$

for  $q > 0$ , such that  $E|\tilde{Z}^\kappa(1)|^q < \infty$ , where  $\tau_0$  is an arbitrary constant, and  $a$  is an analysis scale such that  $a\tau_0 \leq 1$ . The scaling exponents  $\zeta(q)$  are given in (49). To reproduce such behaviors from a single realization of  $\tilde{Z}^\kappa(t)$ , one usually replaces the ensemble averages with time averages, called structure functions,

$$S(a, q) = \frac{1}{n_a} \sum_{k=0}^{n_a} |\tilde{Z}^\kappa(k a \tau_0 + a \tau_0) - \tilde{Z}^\kappa(k a \tau_0)|^q$$

where  $n_a$  is the number of such increments available at scale  $a$ . The estimator  $\hat{\zeta}(q)$  of the scaling exponent  $\zeta(q)$  is obtained by performing a weighted linear regression in a  $\log_2 2^j$  against  $\log_2 S(2^j, q)$  diagram. This procedure is fully defined in, for example, Lashermes *et al.* [24].

However, it is of importance to mention that, as now commonly agreed, the time averages  $S(a, q)$  reproduce the power law behaviors  $S(a, q) \sim C''(q) a^{\zeta(q)}$ , implied by (85) above, only over a finite range of  $q \in [0, q_*^+]$ , where  $q_*^+$  is defined as the value of  $q > 1$  such that  $1 + q\zeta'(q) - \zeta(q) = 0$ . This is detailed in Lashermes *et al.* [24].

Fig. 5 shows the mean values of the estimates  $\hat{\zeta}(q)$  for  $q \in [0, q_*^+]$  obtained from averages over 500 realizations of sample

paths of  $\tilde{Z}^\kappa$  (with  $N = 2^{15}$  and  $R = 2^4$ ). As earlier, we take  $\kappa_c = 0.12$ . Results illustrated in Fig. 5 clearly indicate that the estimated scaling exponents  $\hat{\zeta}(q)$  are in fairly satisfactory agreement with the predicted values  $\zeta(q)$  when  $0 < \kappa_c < \kappa$  (case (A)) and  $0 < \kappa < \kappa_c$  (case (B)). They also show that this agreement is less satisfactory when  $\kappa < 0 < \kappa_c$  (case (N)). However, the agreement tends to improve when the oversampling rate  $R$  is increased, at the price, though, of a significant increase in the computational cost and computer memory issues. This clearly indicates that the use of Riemann sums to approximate the continuous sum defining  $Z^\kappa$  is less efficient when  $\kappa < 0$  (compared to when  $\kappa > 0$ ).

The numerical analysis described above can be straightforwardly extended to structure functions  $S^L(2^j, q)$  obtained by replacing increments at scale  $a = 2^j$  by wavelet leaders. Wavelet leaders are defined as local suprema of the discrete wavelet transform coefficients of  $\tilde{Z}^\kappa$  (see Jaffard *et al.* [25], [26] for definitions, implementations, and performance). Such structure functions can be computed both for positive and negative values of  $q \in [q_*^-, q_*^+]$ , where  $q_*^-$  and  $q_*^+$  are the negative and positive zeros of  $1 + q\zeta'(q) - \zeta(q) = 0$ . For multifractal processes, the structure functions  $S^L(2^j, q)$  exhibit power law behaviors in the limit of fine scales  $S^L(2^j, q) \sim c_q 2^{j\zeta^L(q)}$  and a Legendre transform of  $\zeta^L(q)$  provides a tight upper bound of the multifractal spectrum  $D(h)$  of the analyzed process:  $D(h) \leq \min_{q \neq 0} (1 + qh - \zeta^L(q))$  (cf. Jaffard [25]). The  $\zeta^L(q)$  are estimated via a weighted linear fit as above.

Results (cf. Fig. 6) obtained from 500 realizations of  $\tilde{Z}^\kappa$  (with  $N = 2^{15}$  and  $R = 2^4$ ) show a very satisfactory agreements between the estimated scaling exponents  $\hat{\zeta}^L(q)$  and the theoretical exponents  $\zeta(q)$  given in (49) over the full range  $q \in [q_*^-, q_*^+]$ , i.e., including positive and negative values of  $q$ , when  $0 < \kappa_c < \kappa$  (case (A)) and  $0 < \kappa < \kappa_c$  (case (B)). For the case  $\kappa < 0$  (case (N)), a discrepancy is visible, indicating the requirement to use higher oversampling factors.

### F. MRW Versus Subordinated FBM

In this closing subsection, we compare MRW  $Z^\kappa$  and fBm in multifractal time  $V^\kappa$  as defined in (68). We are interested in whether these processes can have the same scaling exponents and, more generally, the same finite-dimensional distributions.

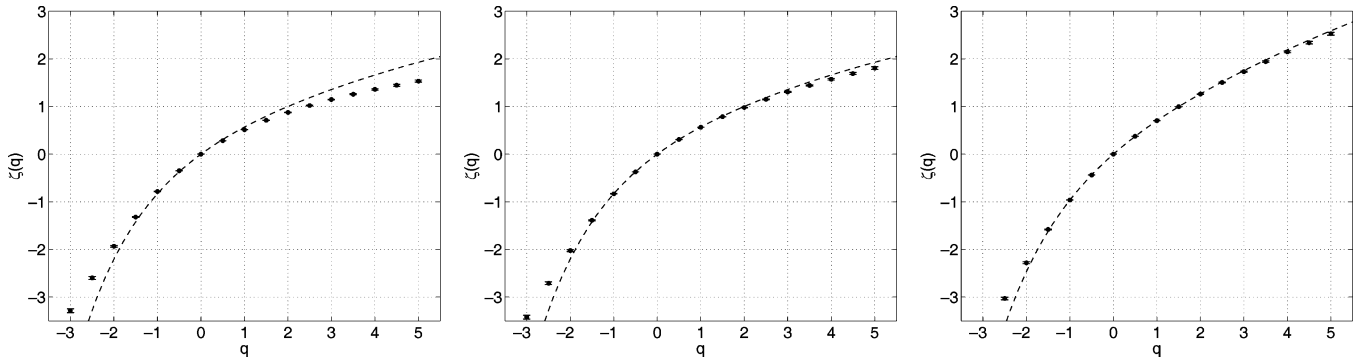


Fig. 6. Scaling exponents from wavelet leaders. For  $\kappa = -0.1 < 0, 0 < \kappa = 0.02 < \kappa_c$ , and  $0 < \kappa_c < \kappa = 0.2$  (from left to right), dashed lines: theoretical scaling exponents; solid points: scaling exponents estimated using wavelet leader based structure functions; for  $q \in [q_*^-, q_*^+]$ . The agreement is very satisfactory when  $\kappa > 0$  for both positive and negative  $q$  while slightly less when  $\kappa < 0$ .

The answer turns out to depend on whether the processes  $Z^\kappa$  and  $V^\kappa$  are defined using the following.

- Case 1: the same parameter  $\kappa$  and function  $\varphi$ ;
- Case 2: the same function  $\varphi$  but different  $\kappa$ ;
- Case 3: the same  $\kappa$  but different functions  $\varphi$ ;
- Case 4: different parameters  $\kappa$  and functions  $\varphi$ .

In Cases 1, 2, and 3,  $Z^\kappa$  and  $V^\kappa$  essentially have different scaling exponents, and hence different finite-dimensional distributions. In Case 4, these processes can have the same scaling exponents. Whether their finite-dimensional distributions are the same is a more challenging issue and will be explored elsewhere. Below is a more detailed treatment of these cases.

*Case 1:* In this case, the scaling exponents of MRW  $Z^\kappa$  and fBm in multifractal time  $V^\kappa$  are

$$\begin{aligned} \zeta_V(q) &= (\kappa + 1/2)q + c\varphi(q(\kappa + 1/2)) \\ \zeta_Z(q) &= (\kappa^* + 1/2)q + c\varphi(q) \end{aligned} \tag{86}$$

where  $\kappa^*$  is defined in (50). When  $\kappa^* = \kappa$ , the scaling exponents are clearly different. When  $\kappa^* = -\varphi(2)/2$ , the same conclusion can be drawn under a mild assumption of twice differentiability of  $\varphi(q)$  at  $q = 0$ . Indeed, if  $\zeta_Z(q) = \zeta_V(q)$ , then  $\zeta_Z''(0) = c(\kappa + 1/2)^2\varphi''(0)$  and  $\zeta_V''(0) = c\varphi''(0)$  have to be equal, which leads to contradiction. If the scaling exponents are different, the corresponding finite-dimensional distributions are different as well.

*Case 2:* This case can be dealt with in the same way as Case 1 assuming twice differentiability of  $\varphi(q)$  at  $q = 0$ .

*Case 3:* In this case, the corresponding scaling exponents are

$$\begin{aligned} \zeta_V(q) &= (\kappa + 1/2)q + c\varphi_1(q(\kappa + 1/2)) \\ \zeta_Z(q) &= (\kappa^* + 1/2)q + c\varphi_2(q) \end{aligned} \tag{87}$$

where  $\kappa^* = \max\{\kappa, \kappa_{2,c}\}$ ,  $\kappa_{2,c} = -\varphi_2(2)/2$ . A particular class of functions  $\varphi_2$  are

$$\varphi_2(q) = \varphi_1(\beta q) - q\varphi_1(\beta) \tag{88}$$

for some  $\beta > 0$ , which are transformations arising in (66). To understand whether the equality of  $\zeta_Z$  and  $\zeta_V$  is possible, several subcases need to be considered.

- If  $\kappa \geq \kappa_{2,c}$ , then  $\kappa^* = \kappa$  and to have  $\zeta_V(q) = \zeta_Z(q)$ , we need  $\varphi_1(q(\kappa + 1/2)) = \varphi_2(q)$ . This is not possible

given basic properties of  $\varphi_i, i = 1, 2$  ( $\varphi_i(0) = \varphi_i(1) = 0$ , concavity).

- If  $\kappa < \kappa_{2,c}$ , on the other hand, then  $\kappa^* = -\varphi_2(2)/2$  and  $\zeta_V(q) = \zeta_Z(q)$  becomes

$$\begin{aligned} (\kappa + \frac{1}{2})q + c\varphi_1(q(\kappa + \frac{1}{2})) \\ = (-\frac{\varphi_2(2)}{2} + \frac{1}{2})q + c\varphi_2(q). \end{aligned} \tag{89}$$

This yields, in particular

$$\begin{aligned} \zeta_V(2) &= (\kappa + \frac{1}{2})2 + c\varphi_1(2(\kappa + \frac{1}{2})) \\ &= (-\frac{\varphi_2(2)}{2} + \frac{1}{2})2 + c\varphi_2(2) = 1. \end{aligned} \tag{90}$$

If  $\zeta_{IDC}(q) = q + c\varphi_1(q)$  are the scaling exponents (28) of the IDC motion, the equality (90) implies that

$$\zeta_{IDC}(2(\kappa + 1/2)) = 1. \tag{91}$$

If  $\kappa \in (-1/2, 0)$ , the relation (91) shows that  $\zeta_{IDC}(q_0) = 1$  for some  $q_0 \in (0, 1)$ . By concavity, one expects that  $\zeta'_{IDC}(1) < 0$  and this situation corresponds to the case when the IDC motion and hence fBm in multifractal time are degenerate (Bacry and Muzy [6]).

If  $\kappa \in (0, \kappa_{2,c})$ , (91) means that  $\zeta_{IDC}(q_0) = 1$  for some  $q_0 \in (1, 2)$ . This corresponds to another unusual situation, when the IDC motion is expected to have infinite variance. When  $\kappa = 0$ , no such conclusion can be drawn. In fact, when  $\kappa = 0$ , (89) is

$$q/2 + c\varphi_1(q/2) = (-\varphi_2(2)/2 + 1/2)q + c\varphi_2(q) \tag{92}$$

or, after noticing that  $\varphi_1(1/2) = -\varphi_2(2)/2$

$$\varphi_2(q) = \varphi_1(q/2) - q\varphi_1(1/2)/2 \tag{93}$$

which is the transformation (88) with  $\beta = 1/2$ . This choice corresponds to using  $\varphi_1$  and  $\beta = 1/2$  as a power in the definition (64) for MRW. As discussed in Section III-B.3, the resulting MRW and fBm in multifractal time have the same finite-dimensional distributions.

Case 4: In this case, the corresponding scaling exponents are

$$\begin{aligned} \zeta_V(q) &= (\kappa_1 + 1/2)q + c\varphi_1(q(\kappa_1 + 1/2)) \\ \zeta_Z(q) &= (\kappa_2^* + 1/2)q + c\varphi_2(q) \end{aligned} \quad (94)$$

where  $\kappa_2^* = \max\{\kappa_2, \kappa_{2,c}\}$ ,  $\kappa_{2,c} = -\varphi_2(2)/2$ . The equality of scaling exponents, in fact, is possible here even in the case when  $\kappa_2^* = \kappa_2$ , that is,  $\kappa_2 \geq -\varphi_2(2)/2$ . Indeed, from a theoretical standpoint, it is just enough to take

$$\kappa_2 = \kappa_1 + \varphi_1(\kappa_1 + 1/2) \quad (95)$$

$$\varphi_2(q) = \varphi_1(q(\kappa_1 + 1/2)) - q\varphi_1(\kappa_1 + 1/2). \quad (96)$$

The following example illustrates that the choice (95)–(96) is legitimate even in the class of log-normal cascades.

Example 4.1: Consider log-normal IDC noises with

$$\varphi_i(q) = \frac{\sigma_i^2}{2}(q - q^2), \quad i = 1, 2. \quad (97)$$

After elementary calculations, the choice (95)–(96) translates into

$$\sigma_2 = H_1\sigma_1, \quad H_2 = H_1 + \frac{\sigma_1^2}{2}(H_1 - H_1^2) \quad (98)$$

where we set  $H_i = \kappa_i + 1/2$  as usual. Note that using (98), the condition  $\kappa_2 > -\varphi_2(2)/2$  is equivalent to

$$\frac{1}{\sigma_1^2} > H_1 \quad (99)$$

given that  $H_1 > 1/2$ . On the other hand, if at least two moments are expected finite for the corresponding IDC motion, it is required (Bacry and Muzy [6]) that

$$\frac{1}{\sigma_1^2} > 1. \quad (100)$$

The latter condition implies (99). Finally, note similarly that  $H_2 < 1$  is equivalent to

$$\frac{2}{\sigma_1^2} > 1 \quad (101)$$

which is even weaker than (99).

Though the choice (95)–(96) yields the same scaling exponents, we expect that the corresponding finite-dimensional distributions of the processes are different. Showing this here, however, would take us beyond the scope of the paper. Just to indicate some of the difficulties, note that the equality of scaling exponents for  $q = 2$  and stationarity of the increments implies that the two processes have the same second-order properties. In order to show that their finite-dimensional distributions are different, higher order properties at several time points (for example,  $EX(s)^2X(t)^2$  where  $X$  is either MRW or fBm in multifractal time) need to be considered. The latter properties go beyond typical multifractal analysis, and will be studied elsewhere.

## V. PROOFS OF AUXILIARY RESULTS

Throughout this section, we shall use the following expressions for the function  $m_r(u)$  defined in (18). In the cases (21) and (22), one has

$$m_r(u) = \begin{cases} 0, & \text{if } u > 1 \\ -c \ln u + c(u - 1), & \text{if } r < u \leq 1 \\ -c \ln r + cu(1 - \frac{1}{r}), & \text{if } 0 \leq u \leq r \end{cases} \quad (102)$$

$$m_r(u) = \begin{cases} 0, & \text{if } u > 1 \\ -c \ln u, & \text{if } r < u \leq 1 \\ -c \ln r + c(1 - \frac{u}{r}), & \text{if } 0 \leq u \leq r \end{cases} \quad (103)$$

respectively.

*Proof of Proposition 2.1:* Suppose for simplicity that  $t = 1$ . For  $\kappa \in (0, 1/2)$ , since  $Q_r(u)$  is nonnegative, it is enough to show that  $E \int_0^1 \int_0^1 Q_r(u)Q_r(v)|u - v|^{2\kappa-1} dudv < \infty$ . This follows from  $E Q_r(u)Q_r(v) \leq C_r < \infty$  for all  $u, v \in [0, 1]$ , using (17) and (102), (103). For  $\kappa \in (-1/2, 0)$ , using Corollary 2.1, it is enough to show that  $I_1 < \infty$  and  $I_2 < \infty$ , where

$$I_1 = E \int_0^1 (Q_r(u)(t - u)^\kappa)^2 du \quad (104)$$

$$I_2 = E \int_0^1 u^{-2\kappa} du \left( \int_0^1 |s^\kappa Q_r(s) - u^\kappa Q_r(u)|(u - s)_+^{\kappa-1} ds \right)^2. \quad (105)$$

The case of  $I_1$  is immediate since  $E Q_r(u)^2 = C_r < \infty$  and  $2\kappa + 1 > 0$ . For  $I_2$ , by the generalized Minkowski's inequality, it is enough to show that  $J_1 < \infty$  and  $J_2 < \infty$ , where

$$J_1 = \int_0^r u^{-2\kappa} \left( \int_0^u \frac{G(u, s)^{1/2}}{(u - s)^{1-\kappa}} ds \right)^2 du \quad (106)$$

$$J_2 = \int_r^1 u^{-2\kappa} \left( \int_{u-r}^u \frac{G(u, s)^{1/2}}{(u - s)^{1-\kappa}} ds \right)^2 du \quad (107)$$

with

$$\begin{aligned} G(u, s) &= E(s^\kappa Q_r(s) - u^\kappa Q_r(u))^2 \\ &= (s^{2\kappa} + u^{2\kappa})e^{-\varphi(2)m_r(0)} - 2s^\kappa u^\kappa e^{-\varphi(2)m_r(u-s)} \end{aligned} \quad (108)$$

and  $m_r(u)$  defined in (18). Then, after a change of variables  $s = uv$ , write  $J_1$  as

$$J_1 = C_r \int_0^r u^{-2\kappa} \left( \int_0^1 F(u, v) dv \right)^2 du \quad (109)$$

where

$$F(u, v) = \frac{((1 + v^{2\kappa}) - 2v^{2\kappa}e^{-\varphi(2)(m_r(u-uv) - m_r(0))})^{1/2}}{(1 - v)^{1-\kappa}}. \quad (110)$$

We have  $J_1 < \infty$  if  $J_{1,1} < \infty$  and  $J_{1,2} < \infty$ , where

$$J_{1,1} = \int_0^r u^{2\kappa} du \left( \int_0^1 \frac{((1 + v^{2\kappa}) - 2v^{2\kappa})^{1/2}}{(1 - v)^{1-\kappa}} dv \right)^2 \quad (111)$$

$$J_{1,2} = \int_0^r u^{2\kappa} \left( \int_0^1 \frac{(m_r(u(1 - v)) - m_r(0))^{1/2}}{(1 - v)^{1-\kappa}} dv \right)^2 du. \quad (112)$$

Finiteness of  $J_{1,1}$  follows from  $\kappa \in (-1/2, 1/2)$  and  $(1 + v^{2\kappa}) - 2v^{2\kappa} \sim C(1 - v)$ , as  $v \uparrow 1$ . Finiteness of  $J_{1,2}$  follows from (102), (103) by observing that  $m_r(0) - m_r(u(1 - v)) \leq C_r u(1 - v)$ . This shows that  $J_1 < \infty$ . The finiteness of  $J_2$  can be proved in a similar way.  $\square$

In the proof of Proposition 3.1 below, we shall use the following notation and facts. Let  $\mathcal{P}_{2p}$  consist of collections of pairs  $\{(i, j)\}$  such that  $i < j$ , and

$$(i, j) \cap (i', j') = \emptyset$$

for two pairs  $(i, j)$  and  $(i', j')$  from the same collection, and

$$\bigcup_{\{(i,j)\}} \{i, j\} = \{1, 2, \dots, 2p\}.$$

For example,  $\mathcal{P}_2$  consists of  $\{(1, 2)\}$ , and  $\mathcal{P}_4$  consists of  $\{(1, 2), (3, 4)\}, \{(1, 3), (2, 4)\}, \{(1, 4), (2, 3)\}$ . Let also  $\mathcal{P}_{2p-1}$  consists of all pairs  $\{(i, j)\} = \{(i', j')\} \setminus \{(i'_{2p}, 2p)\}$  with  $\{(i', j')\} \in \mathcal{P}_{2p}$  and  $(i'_{2p}, 2p)$  indicating a pair containing  $2p$ . For example,  $\mathcal{P}_3$  consists of  $\{(1, 2)\}, \{(1, 3)\}, \{(2, 3)\}$ . We shall also regularly use the fact that

$$\sum_{2 \leq k \leq l \leq q} \alpha_{l-k+2} = \varphi(q), \tag{113}$$

where  $\alpha_k$  is defined in (20) (see also (21) in Bacry and Muzy [6]).

*Proof of Proposition 3.1:* We suppose for simplicity that  $t = 1$  and consider only the case of exact scale invariance (22). (It is clear from the proof below that the arguments can also be adapted to the case (21).) Observe that, by conditioning on  $Q_r$  and using (37) and (31)

$$\begin{aligned} E|Z_r^\kappa(1)|^{2p} &= CE \left( \int_0^1 \int_0^1 Q_r(u)Q_r(v)|u - v|^{2\kappa-1} dudv \right)^p \\ &= C \underbrace{\int_0^1 \dots \int_0^1}_{2p} EQ_r(u_1)Q_r(v_1) \dots Q_r(u_p)Q_r(v_p) \\ &\quad |u_1 - v_1|^{2\kappa-1} \dots |u_p - v_p|^{2\kappa-1} du_1 dv_1 \dots du_p dv_p. \end{aligned}$$

(Here and below,  $C$  will denote a generic constant that may change from line to line.) By using (19), we further obtain (with the notation  $\mathcal{P}_{2p}$  above) that

$$\begin{aligned} E|Z_r^\kappa(1)|^{2p} &= C \sum_{\{(i,j)\} \in \mathcal{P}_{2p}} \int_{0 < z_1 < \dots < z_{2p} < 1} dz_1 \dots dz_{2p} \\ &\quad \exp \left\{ - \sum_{1 \leq k < l \leq 2p} \alpha_{l-k+1} m_r(z_l - z_k) \right\} \prod_{\{(i,j)\}} (z_j - z_i)^{2\kappa-1}. \end{aligned} \tag{114}$$

Note that, after a change of variables  $z_1 = w_1, z_k - z_{k-1} = w_k, k = 2, \dots, 2p$ , we also have

$$E|Z_r^\kappa(1)|^{2p} \leq C \sum_{\{(i,j)\} \in \mathcal{P}_{2p}} \int_0^1 \dots \int_0^1 dw_2 \dots dw_{2p}$$

$$\begin{aligned} &\exp \left\{ - \sum_{2 \leq k \leq l \leq 2p} \alpha_{l-k+2} m_r(w_k + \dots + w_l) \right\} \\ &\prod_{\{(i,j)\}} (w_{i+1} + \dots + w_j)^{2\kappa-1} =: CI_r(2p) \end{aligned} \tag{115}$$

where the inequality above is due to dropping the restrictions  $w_1 + \dots + w_k < 1, k = 2, \dots, 2p$ . We will show that

$$I_r(2p) \sim Cn(r)^{2p} \tag{116}$$

where  $n(r)$  is given by (53). Since  $E|Z_r^\kappa(1)|^{2p}$  increases as  $r \rightarrow 0$  in view of (115), this immediately yields (52) in case (A). In cases (E) and (B), similar arguments to those below would show that the rate of increase of the difference between  $I_r(2p)$  and  $E|Z_r^\kappa(1)|^{2p}$  is slower than  $n(r)^{2p}$ . Moreover, only the case (B) will be considered. We prove (116) for cases (A) and (B) separately.

*Case (A):* Since  $m_r(u) \leq -c \ln u$  for all  $u, r > 0$ , it is enough to show that  $I(2p) < \infty$ , where

$$\begin{aligned} I(q) &= \int_0^1 \dots \int_0^1 dw_2 \dots dw_q \\ &\prod_{2 \leq k \leq l \leq q} (w_k + \dots + w_l)^{c\alpha_{l-k+2}} \prod_{\{(i,j)\}} (w_{i+1} + \dots + w_j)^{2\kappa-1} \end{aligned} \tag{117}$$

and  $\{(i, j)\}$  is any fixed collection of pairs from  $\mathcal{P}_{2p}$ . This will be achieved through induction in  $p$  and we shall also use  $I(q)$  for odd powers  $q$  defined by (117) but where  $\{(i, j)\} \in \mathcal{P}_q$ .

When  $p = 1, I(2) = \int_0^1 w_2^{c\alpha_2+2\kappa-1} dw_2$  which is finite since  $c\alpha_2 + 2\kappa = c\varphi(2) + 2\kappa > 0$  in case (A). Suppose  $I(k) < \infty, k = 1, \dots, q-1$ , and consider the case of  $I(q)$ . (Note that, since  $\varphi$  is concave,  $(\kappa + 1/2)2 + c\varphi(2) > 1$  and  $(\kappa + 1/2)q + c\varphi(q) > 1$  imply  $(\kappa + 1/2)k + c\varphi(k) > 1$  for  $k = 2, \dots, q$ .) Denote by  $I_n(q), n = 2, \dots, q$ , the integral  $I(q)$  restricted to the set  $w_m < w_n, \text{ all } m \neq n$ . It is then enough to show that

$$I_n(q) < \infty, \quad n = 2, \dots, q. \tag{118}$$

We need to consider the cases  $q = 2p$  and  $q = 2p-1$  separately.

Consider first the case of even  $q = 2p$ . Then, making a change of variables  $w_n = s_n, w_m = s_n t_m, m \neq n$ , and using the fact (113), we obtain that

$$\begin{aligned} I_n(2p) &= \int_0^1 ds_n s_n^{(2\kappa-1)p+2p-2+c\varphi(2p)} \\ &\quad \int_0^1 \dots \int_0^1 dt_2 \dots dt_{n-1} dt_{n+1} \dots dt_{2p} \\ &\quad \prod_{2 \leq k \leq l \leq 2p} (t_k + \dots + t_l)^{c\alpha_{l-k+2}} \prod_{\{(i,j)\}} (t_{i+1} + \dots + t_j)^{2\kappa-1} \end{aligned} \tag{119}$$

where  $t_n = 1$  by convention in the last multiple integral. Since  $t_n = 1$ , we further have

$$\begin{aligned} I_n(2p) &\leq C \int_0^1 ds_n s_n^{c(2p)-2} \int_0^1 \dots \int_0^1 dt_2 \dots dt_{n-1} \\ &\quad \prod_{2 \leq k \leq l \leq n-1} (t_k + \dots + t_l)^{c\alpha_{l-k+2}} \prod_{\{(i',j')\}} (t_{i'+1} + \dots + t_{j'})^{2\kappa-1} \end{aligned}$$

$$\begin{aligned} & \times \int_0^1 \cdots \int_0^1 dt_{n+1} \cdots dt_{2p} \\ & \prod_{2 \leq k \leq l \leq 2p-(n-1)} (t_{n-1+k} + \cdots + t_{n-1+l})^{c\alpha_l - k + 2} \\ & \prod_{\{(i^*, j^*)\}} (t_{n+i^*} + \cdots + t_{n+j^*})^{2\kappa-1} \end{aligned} \quad (120)$$

where  $\{(i', j')\}$  is a collection of pairs from  $\mathcal{P}_{n-1}$  and  $\{(i^*, j^*)\}$  is a collection of pairs from  $\mathcal{P}_{2p-(n-1)}$ . These collections may not be defined uniquely and can be chosen only to make a convenient bound. For example, in the case  $p = 2$ ,  $\{(i, j)\} = \{(2, 3), (1, 4)\}$ , and  $n = 3$ ,  $I_3(4)$  in (119) is

$$I_3(4) = \int_0^1 ds_3 s_3^{(2\kappa+1)2+c\varphi(4)-2} \int_0^1 \int_0^1 dt_2 dt_4 t_2^{c\alpha_2(1+t_2)+c\alpha_3(1+t_4)-2} t_4^{c\alpha_4(1+t_4)-2} \quad (121)$$

which is bounded in the argument above as

$$I_3(4) \leq C \int_0^1 ds_3 s_3^{\zeta(4)-2} \int_0^1 dt_2 t_2^{c\alpha_2+2\kappa-1} \int_0^1 dt_4 t_4^{c\alpha_4+2\kappa-1}. \quad (122)$$

(Note throughout that  $2\kappa - 1 < 0$  and  $\alpha_k \leq 0$ .) Hence

$$I_n(2p) \leq C \int_0^1 ds_n s_n^{\zeta(2p)-2} I(n-1) I(2p-(n-1))$$

which is bounded by induction, and since  $\zeta(2p) > 1$ . The case of odd  $q = 2p - 1$  can be considered similarly and involves the integral

$$\int_0^1 ds_n s_n^{(2\kappa-1)(p-1)+2p-3+c\varphi(2p-1)}.$$

This integral is finite since

$$(2\kappa - 1)(p - 1) + 2p - 3 + c\varphi(2p - 1) > (\kappa + 1/2)(2p - 1) + c\varphi(2p - 1) - 1 > 0.$$

*Case (B):* We will show that  $r^{-(c\varphi(2)+2\kappa)p} J_r(p) \sim C$ , where

$$\begin{aligned} J_r(p) &= \int_0^1 \cdots \int_0^1 dw_2 \cdots dw_{2p} \\ & \exp \left\{ - \sum_{2 \leq k \leq l \leq 2p} \alpha_{l-k+2} m_r(w_k + \cdots + w_l) \right\} \prod_{k=1}^p w_{2k}^{2\kappa-1}. \end{aligned} \quad (123)$$

This is one of the terms in (115) with  $\{(i, j)\} = \{(1, 2), (3, 4), \dots, (2p-1, 2p)\}$ . We will argue that the terms in (115) with other  $\{(i, j)\} \in \mathcal{P}_{2p}$  have slower rate of growth than  $r^{(c\varphi(2)+2\kappa)p}$ .

Let  $\tilde{J}_r(p)$  be the integral  $J_r(p)$  restricted to  $w_2 > r, \dots, w_{2p} > r$ . We will show first that  $\tilde{J}_r(p) \sim r^{(c\varphi(2)+2\kappa)p} C$  and then argue that the same happens with  $J_r(p)$ . Observe that

$$\begin{aligned} \tilde{J}_r(p) &= \int_r^1 \cdots \int_r^1 dw_2 \cdots dw_{2p} \\ & \prod_{2 \leq k \leq l \leq 2p} (w_k + \cdots + w_l)^{c\alpha_l - k + 2} \prod_{k=1}^p w_{2k}^{2\kappa-1}. \end{aligned} \quad (124)$$

Making a change of variables  $w_{2k} = rv_{2k}$ ,  $k = 1, \dots, p$ ,  $w_{2k+1} = u_{2k+1}$ ,  $k = 1, \dots, p-1$ , we obtain that

$$\begin{aligned} \tilde{J}_r(p) &= r^{(c\alpha_2+2\kappa)p} \\ & \int_1^{r^{-1}} \int_1^{r^{-1}} \cdots \int_1^{r^{-1}} dv_2 dv_4 \cdots dv_{2p} \prod_{k=1}^p v_{2k}^{c\alpha_2+2\kappa-1} \\ & \int_r^1 \int_r^1 \cdots \int_r^1 du_3 du_5 \cdots du_{2p-1} \\ & \prod_{2 \leq k \leq l \leq 2p} (u_k + \cdots + u_l)^{c\alpha_l - k + 2} \end{aligned} \quad (125)$$

where  $u_{2k} = rv_{2k}$  by convention. Then, by dominated convergence theorem

$$\tilde{J}_r(p) \sim r^{(c\varphi(2)+2\kappa)p} \left( \int_1^{\infty} dv v^{c\varphi(2)+2\kappa-1} \right)^p L(p) \quad (126)$$

as long as  $L(p) < \infty$  (recall also that  $c\varphi(2) + 2\kappa < 0$ ), where

$$L(p) = \int_0^1 \int_0^1 \cdots \int_0^1 du_3 du_5 \cdots du_{2p-1} \prod_{1 \leq k < l \leq p-1} (u_{2k+1} + \cdots + u_{2l+1})^{c\beta_{l-k}} \quad (127)$$

and

$$\beta_m = \alpha_{2m} + 2\alpha_{2m+1} + \alpha_{2m+2}, \quad m = 1, \dots, p-1. \quad (128)$$

Showing that  $L(p) < \infty$  can be done by induction in  $p$ , similarly to and easier than in case (A) above, if

$$c \sum_{1 \leq k < l \leq q} \beta_{l-k} + (q-1) + 1 > 0, \quad q = 2, \dots, p-1.$$

Observe that

$$\begin{aligned} & \sum_{1 \leq k < l \leq q} \beta_{l-k} \\ &= \sum_{2 \leq k \leq l \leq 2q} \alpha_{l-k+2} - (q+1)\alpha_2 = \varphi(2q+2) - (q+1)\varphi(2). \end{aligned}$$

Hence, the condition above becomes

$$(1 - c\varphi(2)) \frac{2q}{2} + c\varphi(2q) > 1, \quad q = 3, \dots, p.$$

Since  $(1 - c\varphi(2)) + c\varphi(2) \geq 1$  and  $\varphi$  is concave, these conditions just follow from  $(1 - c\varphi(2)) \frac{2p}{2} + c\varphi(2p) > 1$ .

The integral  $J_r(p)$  restricted to other regions than that in  $\tilde{J}_r(p)$  can be dealt with in a similar way. For example, the integral  $J_r(p)$  over  $w_2 < r, w_4 < r, \dots, w_{2p} < r$ , and  $w_3 > r, w_5 > r, \dots, w_{2p-1} > r$  can be dealt with in exactly the same way as  $\tilde{J}_r(p)$ , and has the rate of growth  $r^{(c\varphi(2)+2\kappa)p}$ . The integral  $J_r(p)$  over  $w_2 < r, w_4 < r, \dots, w_{2p} < r$ , and  $w_3 < r, w_5 > r, \dots, w_{2p-1} > r$  has the rate of growth  $o(r^{(c\varphi(2)+2\kappa)p})$  because of the restriction  $w_3 < r$ .

Finally, consider the terms in (115) with  $\{(i, j)\} \in \mathcal{P}_{2p}$  other than  $\{(i, j)\} = \{(1, 2), (3, 4), \dots, (2p-1, 2p)\}$ . These terms can be shown to be of the order  $o(r^{(c\varphi(2)+2\kappa)p})$ . The basic idea is the following. (We will omit further details for shortness



sake.) For example, such term over  $w_k > r, k = 2, \dots, 2p$ , can be bounded by

$$\int_r^1 \dots \int_r^1 dw_2 \dots dw_{2p} \prod_{2 \leq k \leq l \leq 2p} (w_k + \dots + w_l)^{\tilde{\alpha}_{l-k+2}} \prod_{\{k'\}} w_{k'}^{\kappa'+\kappa'-1} \quad (129)$$

where  $\tilde{\alpha}_2 = \alpha_2 = \varphi(2)$ ,  $\tilde{\alpha}_m$ 's satisfy the assumptions of  $\alpha_m$ 's, there are  $p$  different  $k'$  such that  $\{k'\} \subset \{2, \dots, 2p\}$ , and  $\kappa' < \kappa$ . For example, in the case  $p = 2$ , the integral

$$\int_r^1 \int_r^1 \int_r^1 dw_2 dw_3 dw_4 w_2^{c\alpha_2} (w_2 + w_3)^{c\alpha_3} (w_2 + w_3 + w_4)^{c\alpha_4} w_3^{c\alpha_2} (w_3 + w_4)^{c\alpha_3} w_4^{c\alpha_2} (w_2 + w_3)^{2\kappa-1} (w_3 + w_4)^{2\kappa-1} \quad (130)$$

is bounded by

$$\int_r^1 \int_r^1 \int_r^1 dw_2 dw_3 dw_4 w_2^{c\alpha_2+\kappa+\kappa'-1} (w_2 + w_3)^{c\alpha_3+(\kappa-\kappa')} (w_2 + w_3 + w_4)^{c\alpha_4} w_3^{c\alpha_2} (w_3 + w_4)^{c\alpha_3+(\kappa-\kappa')} w_4^{c\alpha_2+\kappa+\kappa'-1}. \quad (131)$$

When  $\kappa'$  is close enough to  $\kappa, \kappa' < \kappa$ , the exponents  $\tilde{\alpha}_2 = \alpha_2, \tilde{\alpha}_3 = \alpha_3 + \kappa - \kappa', \tilde{\alpha}_4 = \alpha_4$  have properties of the original  $\alpha_2, \alpha_3, \alpha_4$ . Because of the presence of  $\kappa' < \kappa$ , an argument similar to that above would show that (129) is of the order  $O(r^{(c\varphi(2)+2\kappa)p})$ .  $\square$

*Proof of Proposition 3.2:* We consider the simpler case of the exact scale invariance (22), and only  $t = 1$ . Observe that, by conditioning on  $Q_r$  and using (33) and (38)

$$\begin{aligned} & \frac{EZ_r^\kappa(1)^2}{C_\kappa \Gamma(\kappa+1)^2} \\ &= E \int_0^1 \left\{ s^{-\kappa} \int_s^1 F(u, s) du + Q_r(s)(1-s)^\kappa \right\}^2 ds \\ &= \kappa^2 E \int_0^1 s^{-2\kappa} \left\{ \int_s^1 F(u, s) du \right\}^2 ds \\ & \quad + 2\kappa E \int_0^1 s^{-\kappa} Q_r(s)(1-s)^\kappa \left\{ \int_s^1 F(u, s) du \right\} ds \\ & \quad + E \int_0^1 Q_r(s)^2 (1-s)^{2\kappa} ds \\ &=: \kappa^2 T_{1,r} + 2\kappa T_{2,r} + T_{3,r} \end{aligned} \quad (132)$$

where

$$F(u, s) = \left( u^\kappa Q_r(u) - s^\kappa Q_r(s) \right) (u-s)^{\kappa-1}. \quad (133)$$

To prove (54), we will show that  $T_{1,r}$  diverges faster than  $T_{2,r}$  and  $T_{3,r}$ , and find the corresponding rate of divergence of  $T_{1,r}$ . We first examine  $T_{3,r}$  and  $T_{2,r}$ , and then turn to  $T_{1,r}$ .

The case of  $T_{3,r}$  is simple. By using (16) and (103), we have

$$\begin{aligned} T_{3,r} &= e^{-c\varphi(2)} r^{c\varphi(2)} \int_0^1 (1-s)^{2\kappa} ds \\ &= \frac{e^{-c\varphi(2)} r^{c\varphi(2)}}{2\kappa+1} = O(r^{c\varphi(2)}). \end{aligned} \quad (134)$$

Turning to  $T_{2,r}$ , we have

$$\begin{aligned} T_{2,r} &= \int_{0 < s < u < 1} (1_{\{u-s>r\}} + 1_{\{u-s<r\}}) s^{-\kappa} (1-s)^\kappa \\ & \quad \left( u^\kappa EQ_r(u)Q_r(s) - s^\kappa EQ_r(s)^2 \right) (u-s)^{\kappa-1} ds du \\ &=: T_{2,r}^{(1)} + T_{2,r}^{(2)}. \end{aligned} \quad (135)$$

By using (17) and (103)

$$\begin{aligned} T_{2,r}^{(1)} &= \int_{0 < s < u < 1, u-s > r} s^{-\kappa} (1-s)^\kappa \left( u^\kappa (u-s)^{c\varphi(2)} \right. \\ & \quad \left. - s^\kappa e^{-c\varphi(2)} r^{c\varphi(2)} \right) (u-s)^{\kappa-1} duds \\ &= I_1 - r^{c\varphi(2)} e^{-c\varphi(2)} I_2 \end{aligned} \quad (136)$$

and, since  $\kappa < 0$

$$\begin{aligned} I_1 &= \int_{0 < s < u < 1, u-s > r} s^{-\kappa} (1-s)^\kappa u^\kappa (u-s)^{c\varphi(2)+\kappa-1} duds \\ &\leq \int_{0 < s < u < 1, u-s > r} (1-s)^\kappa (u-s)^{c\varphi(2)+\kappa-1} duds \\ &= \frac{1}{c\varphi(2)+\kappa} \left( \frac{1-r^{c\varphi(2)+2\kappa+1}}{c\varphi(2)+2\kappa+1} - r^{c\varphi(2)+\kappa} \frac{1-r^{\kappa+1}}{\kappa+1} \right) \\ &= O(r^{c\varphi(2)+\kappa}), \end{aligned} \quad (137)$$

and

$$\begin{aligned} I_2 &= \int_{0 < s < u < 1, u-s > r} (1-s)^\kappa (u-s)^{\kappa-1} duds \\ &= \frac{1}{\kappa} \left( \frac{1-r^{2\kappa+1}}{2\kappa+1} - \frac{r^\kappa(1-r^{\kappa+1})}{\kappa+1} \right) = O(r^\kappa). \end{aligned} \quad (138)$$

Thus, by collecting the above expressions

$$T_{2,r}^{(1)} = O(r^{c\varphi(2)+\kappa}). \quad (139)$$

By using (17) and (103)

$$\begin{aligned} T_{2,r}^{(2)} &= e^{-c\varphi(2)} r^{c\varphi(2)} \int_{0 < s < u < 1, u-s < r} ds du \\ & \quad s^{-\kappa} (1-s)^\kappa \frac{\left( u^\kappa e^{c\varphi(2)} \frac{u-s}{r} - s^\kappa \right)}{(u-s)^{1-\kappa}} \\ &= -e^{-c\varphi(2)} r^{c\varphi(2)} (J_1 + J_2). \end{aligned} \quad (140)$$

Here

$$\begin{aligned} J_1 &= \int_{0 < s < u < 1, u-s < r} s^{-\kappa} (1-s)^\kappa \frac{(s^\kappa - u^\kappa) e^{c\varphi(2)} \frac{u-s}{r}}{(u-s)^{1-\kappa}} ds du \\ &\leq \int_0^r dw w^{\kappa-1} \int_0^{1-w} s^{-\kappa} (1-s)^\kappa (s^\kappa - (s+w)^\kappa) ds. \end{aligned} \quad (141)$$

Observe that, with  $\varepsilon < 1$

$$\begin{aligned} 0 \leq s^\kappa - (s+w)^\kappa &= -\kappa \int_s^{s+w} u^{\kappa-1} du \\ &\leq -\kappa s^{\kappa-1+\varepsilon} \int_s^{s+w} u^{-\varepsilon} du \leq -\frac{\kappa}{1-\varepsilon} s^{\kappa-1+\varepsilon} w^{1-\varepsilon}. \end{aligned} \quad (142)$$

Hence, with  $\kappa + 1 < \varepsilon < 1$

$$J_1 \leq \frac{(-\kappa)}{1-\varepsilon} \int_0^r w^{\kappa-\varepsilon} dw \int_0^1 s^{-1+\varepsilon} (1-s)^\kappa ds = O(r^{\kappa-\varepsilon+1}).$$

Similarly

$$\begin{aligned} J_2 &= \int_{0 < s < u < 1, u-s < r} (1-s)^\kappa \kappa \frac{(1 - e^{c\varphi(2)\frac{u-s}{r}})}{(u-s)^{1-\kappa}} ds du \\ &\leq -\frac{c\varphi(2)}{r} \int_{0 < s < u < 1, u-s < r} (1-s)^\kappa (u-s)^\kappa ds du \\ &\leq -\frac{c\varphi(2)}{(\kappa+1)^2} r^\kappa = O(r^\kappa). \end{aligned} \quad (143)$$

Collecting the above results yields

$$T_{2,r}^{(2)} = O(r^{c\varphi(2)+\kappa}) \quad (144)$$

and the relations (135), (139), and (144) show that

$$T_{2,r} = O(r^{c\varphi(2)+\kappa}). \quad (145)$$

Consider now the term  $ET_{1,r}$  in (132). Let

$$F(u) = \begin{cases} |u|^{c\varphi(2)}, & \text{if } |u| > 1 \\ e^{-c\varphi(2)} e^{c\varphi(2)|u|}, & \text{if } |u| < 1. \end{cases} \quad (146)$$

By making a change of variables  $s = rz, u-s = rx, v-s = ry$ , and using (17) and (103), we have

$$\begin{aligned} \frac{T_{1,r}}{2} &= \int_{0 < s < u < v < 1} s^{-2\kappa} (u^\kappa v^\kappa EQ_r(u)Q_r(v) \\ &\quad - u^\kappa s^\kappa EQ_r(u)Q_r(s) - v^\kappa s^\kappa EQ_r(v)Q_r(s) \\ &\quad + s^{2\kappa} EQ_r(s)^2) (u-s)^{\kappa-1} (v-s)^{\kappa-1} ds du dv \\ &= r^{2\kappa+c\varphi(2)} \int_0^{r^{-1}} dz \int_0^{r^{-1}-z} dx \int_x^{r^{-1}-z} dy \\ &\quad \left( \left(1 + \frac{x}{z}\right)^\kappa \left(1 + \frac{y}{z}\right)^\kappa F(y-x) - \left(1 + \frac{x}{z}\right)^\kappa F(x) \right. \\ &\quad \left. - \left(1 + \frac{y}{z}\right)^\kappa F(y) + F(0) \right) x^{\kappa-1} y^{\kappa-1} \end{aligned} \quad (147)$$

so that

$$\begin{aligned} \frac{T_{1,r}}{2} &\sim r^{2\kappa+c\varphi(2)} \int_{0 < x < y} dx dy \\ &\quad (F(y-x) - F(x) - F(y) + F(0)) x^{\kappa-1} y^{\kappa-1}. \end{aligned} \quad (148)$$

The result (54) follows from (134), (145), and (148), since  $\kappa < 0$ .  $\square$

*Proof of Lemma 3.1:* In the case  $\kappa \in (0, 1/2)$

$$\begin{aligned} EZ_r^\kappa(t)Z_{r^*}^\kappa(t) &= C_\kappa \int_0^t \int_0^t EQ_r(u)Q_{r^*}(v)|u-v|^{2\kappa-1} dudv \\ &= C_\kappa \int_0^t \int_0^t EQ_r(u)Q_r(v)|u-v|^{2\kappa-1} dudv \\ &= EZ_r^\kappa(t)^2 \end{aligned}$$

since  $\{Q_r(v)\}_{r>0}$  is a martingale (Lemma 3 in Chainais *et al.* [9]). The case  $\kappa \in (-1/2, 0)$  can be proved in a similar way using the expression of the type (132).  $\square$

*Proof of Theorem 3.3:* Using Theorem 4.2.3 in Kwapień and Woyczyński [27], it is enough to show that

- (i)  $\{Z_{1/T}^\kappa(t)\}_{t \in [0,1]}$  is a martingale in  $T \geq 1$ ;
- (ii)  $\{Z_{1/T}^\kappa(t)\}_{t \in [0,1]}$  converges in distribution;
- (iii)  $\sup_{T \geq 1} E \sup_{t \in [0,1]} |Z_{1/T}^\kappa(t)| < \infty$ .

The martingale property in (i) is equivalent to  $\{Z_{1/T}^\kappa(t), T \geq 1\}$  being a martingale for fixed  $t$  (but with respect to the same filtration for all  $t$ ). The proof of (i)–(iii) is given next.

(i) Consider a filtration  $\mathcal{F} = \{\mathcal{F}_T, T \geq 1\}$  of  $\sigma$ -fields given by  $\mathcal{F}_T = \sigma\{M(A), A \in \mathcal{B}([1/T, \infty) \times \mathbb{R})\} \vee \sigma\{B^\kappa(t), t \in \mathbb{R}\}$ , where  $\mathcal{B}$  indicates Borel  $\sigma$ -field. The  $\sigma$ -field  $\mathcal{F}_S$  is generated by smooth variables of the form

$$\Phi = F(M(A_1), \dots, M(A_n)) \exp\left\{i \sum_{j=1}^m \beta_j B^\kappa(t_j)\right\}$$

where  $m, n \in \mathbb{N} \cup \{0\}$ ,  $F : \mathbb{R}^n \mapsto \mathbb{R}$  is a bounded, continuous function  $\beta_j, t_j \in \mathbb{R}$ , and  $A_j \in \mathcal{B}([1/S, \infty) \times \mathbb{R})$ . For the martingale property, it is then enough to show that  $E\Phi Z_{1/T}^\kappa(t) = E\Phi Z_{1/S}^\kappa(t)$  for  $T \geq S \geq 1$ .

Letting  $E^M$  be the expectation conditionally on  $M$ , observe that  $(Z_{1/T}^\kappa(t), B^\kappa(t_1), \dots, B^\kappa(t_m))$  is a conditionally Gaussian vector with the covariance structure consisting, in particular, of

$$E^M Z_{1/T}^\kappa(t) B^\kappa(t_j) = C_\kappa \int_0^t \int_0^{t_j} Q_{1/T}(u)|u-v|^{2\kappa-1} dudv.$$

Using the fact that, for a Gaussian vector  $(X_1, \dots, X_{m+1})$  with covariance matrix  $(c_{k,l})_{k,l=1, \dots, m+1}$

$$EX_1 \exp\left\{i \sum_{j=2}^{m+1} \beta_j X_j\right\} = -i \sum_{j=2}^{m+1} \beta_j c_{1,j} E \exp\left\{i \sum_{j=2}^{m+1} \beta_j X_j\right\}$$

this yields that

$$\begin{aligned} E\Phi Z_{1/T}^\kappa(t) &= -i C_\kappa \sum_{j=1}^m E\left(F(M(A_1), \dots, M(A_n)) \right. \\ &\quad \left. \int_0^t \int_0^{t_j} Q_{1/T}(u)|u-v|^{2\kappa-1} dudv \exp\left\{i \sum_{j=1}^m \beta_j B^\kappa(t_j)\right\}\right). \end{aligned}$$

Using the martingale property of  $\{Q_{1/T}(u), T \geq 1\}$  and conditioning on  $\mathcal{F}_S$ , the last expression is equal to  $E\Phi Z_{1/S}^\kappa(t)$ .

(ii) By Proposition 3.1, the collection  $\{Z_{1/T}^\kappa(t), T \geq 1\}$  is bounded in  $L^2(\Omega)$ . Then, since  $\{Z_{1/T}^\kappa(t), T \geq 1\}$  is a martingale by (i), it converges almost surely. The almost sure convergence also implies that of finite-dimensional distributions. For

the convergence in distribution in the space of continuous functions  $C([0, 1], \mathbb{R})$ , since  $c\varphi(2) + 2\kappa > 0$ , it is therefore enough to show (Billingsley [28]) that, for all  $T \geq 1, 0 \leq t_1 \leq t_2 \leq 1$

$$E(Z_{1/T}^\kappa(t_2) - Z_{1/T}^\kappa(t_1))^2 \leq C(t_2 - t_1)^{c\varphi(2)+2\kappa+1}.$$

For this, use the fact  $EQ_{1/T}(u)Q_{1/T}(v) \leq C|u - v|^{c\varphi(2)}$  for all  $T \geq 1$ , to write

$$\begin{aligned} E(Z_{1/T}^\kappa(t_2) - Z_{1/T}^\kappa(t_1))^2 &\leq c \int_{t_1}^{t_2} \int_{t_1}^{t_2} |u - v|^{c\varphi(2)+2\kappa-1} dudv \\ &\leq c'(t_2 - t_1)^{c\varphi(2)+2\kappa+1}. \end{aligned} \tag{149}$$

(iii) For this part, observe that

$$\begin{aligned} E \sup_{t \in [0,1]} |Z_{1/T}^\kappa(t)| &= \int_0^\infty P(\sup_{t \in [0,1]} |Z_{1/T}^\kappa(t)| > x) dx \\ &= E \int_0^\infty P(\sup_{t \in [0,1]} |Z_{1/T}^\kappa(t)| > x | Q_{1/T}) dx. \end{aligned} \tag{150}$$

Conditionally on  $Q_{1/T}$ ,  $Z_{1/T}^\kappa(t)$  is a Gaussian continuous process in  $t$ . By the Borell inequality for continuous Gaussian processes (Adler [29, p. 43, relation (2.6)]), we have

$$\begin{aligned} P(\sup_{t \in [0,1]} |Z_{1/T}^\kappa(t)| > x | Q_{1/T}) &\leq 2 \exp \left\{ -\frac{1}{2} (x - E(\sup_{t \in [0,1]} |Z_{1/T}^\kappa(t)| | Q_{1/T}))^2 / \sigma_T^2 \right\} \end{aligned} \tag{151}$$

where

$$\sigma_T^2 = \sup_{t \in [0,1]} E(|Z_{1/T}^\kappa(t)|^2 | Q_{1/T}). \tag{152}$$

Substituting (151) into (150), extending the integration range to  $\mathbb{R}$ , and making a change of variables  $x - E(\sup_{t \in [0,1]} |Z_{1/T}^\kappa(t)| | Q_{1/T}) = y$  yields

$$\begin{aligned} E \sup_{t \in [0,1]} |Z_{1/T}^\kappa(t)| &\leq 2E \int_{\mathbb{R}} e^{-\frac{y^2}{2\sigma_T^2}} dy = 2\sqrt{2\pi} E\sigma_T \\ &\leq 2\sqrt{2\pi} (E\sigma_T^2)^{1/2}. \end{aligned} \tag{153}$$

Since  $Q_{1/T}(u) > 0$ , observe now that

$$\begin{aligned} \sigma_T^2 &= C_\kappa \sup_{t \in [0,1]} \int_0^t \int_0^t Q_{1/T}(u)Q_{1/T}(v) |u - v|^{2\kappa-1} dudv \\ &= C_\kappa \int_0^1 \int_0^1 Q_{1/T}(u)Q_{1/T}(v) |u - v|^{2\kappa-1} dudv. \end{aligned}$$

Hence,  $E\sigma_T^2 = EZ_{1/T}^\kappa(1)^2$  and substituting this into (153), we obtain that

$$\sup_{T \geq 1} E \sup_{t \in [0,1]} |Z_{1/T}^\kappa(t)| \leq 2\sqrt{2\pi} (\sup_{T \geq 1} EZ_{1/T}^\kappa(1)^2)^{1/2}. \tag{154}$$

The conclusion follows since  $\sup_{T \geq 1} EZ_{1/T}^\kappa(1)^2 < \infty$  by Proposition 3.1.  $\square$

*Proof of Theorem 3.4:* We first show (63) and work with  $t = 1$ , for simplicity. Observe that, for  $k \in \mathbb{N}$

$$\begin{aligned} E|Z_r^\kappa(1)|^q &= C_{\kappa,q} E \left( \int_0^1 \int_0^1 Q_r(u)Q_r(v) |u - v|^{2\kappa-1} dudv \right)^{q/2} \\ &\leq C_{\kappa,q} (I_1 + I_2) \end{aligned} \tag{155}$$

where

$$\begin{aligned} I_1 &= E \left( \int_0^1 \int_0^1 1_{A_1}(u, v) Q_r(u)Q_r(v) |u - v|^{2\kappa-1} dudv \right)^{q/2} \\ I_2 &= E \left( \int_0^1 \int_0^1 1_{A_2}(u, v) Q_r(u)Q_r(v) |u - v|^{2\kappa-1} dudv \right)^{q/2} \end{aligned}$$

where  $A_1 = \{(u, v) : |u - v| \leq k^{-1}/2\}$  and  $A_2 = \{(u, v) : |u - v| > k^{-1}/2\}$ . By a change of variables  $u = kx, v = ky$  and relations (23) and (24), we get that

$$I_1 = k^{-\zeta(q)} E \left( \int_0^k \int_0^k F_r(x, y) dx dy \right)^{q/2}$$

where  $r$  is such that  $r \leq k^{-1}$  and

$$F_r(x, y) = 1_{\{|x-y| \leq 1/2\}} Q_{kr}(x)Q_{kr}(y) |x - y|^{2\kappa-1}. \tag{156}$$

Note further that, by stationarity of  $Q_{kr}(\cdot)$

$$\begin{aligned} k^{\zeta(q)} I_1 &\leq \sum_{n=0}^{2k-1} E \left( \int_{n/2}^{(n+1)/2} \int_{n/2}^{(n+1)/2} F_r(x, y) dx dy \right)^{\frac{q}{2}} \\ &\quad + 2 \sum_{n=0}^{2k-2} E \left( \int_{n/2}^{(n+1)/2} \int_{(n+1)/2}^{(n+2)/2} F_r(x, y) dx dy \right)^{\frac{q}{2}} \\ &\leq 2k E \left( \int_0^{1/2} \int_0^{1/2} F_r(x, y) dx dy \right)^{\frac{q}{2}} \\ &\quad + 2(2k - 1) E \left( \int_0^{1/2} \int_{1/2}^1 F_r(x, y) dx dy \right)^{\frac{q}{2}} \\ &\leq 6k E \left( \int_0^1 \int_0^1 F_r(x, y) dx dy \right)^{\frac{q}{2}} \\ &= \frac{6k}{C_{\kappa,q}} E|Z_{kr}^\kappa(1)|^q. \end{aligned} \tag{157}$$

Since  $\{Z_{1/s}^\kappa(s)\}_{s \geq 1}$  is a martingale (see the proof of Theorem 3.3) and  $r \leq rk$

$$I_1 \leq \frac{6k^{1-\zeta(q)}}{C_{\kappa,q}} E|Z_r^\kappa(1)|^q. \tag{158}$$

On the other hand

$$I_2 \leq (2k)^{(1-\kappa/2)q} E \left( \int_0^1 Q_r(u) du \right)^q. \tag{159}$$

If  $k$  is such that  $6k^{1-\zeta(q)} < 1$ , we get from (155), (158), and (159) that

$$E|Z_r^\kappa(1)|^q \leq \frac{(2k)^{(1-\kappa/2)q}}{1 - 6k^{1-\zeta(q)}} E \left( \int_0^1 Q_r(u) du \right)^q \tag{160}$$

for all  $r \leq k^{-1}$ . Since  $q + c\varphi(q) > (\kappa_c + 1/2)q + c\varphi(q) = \zeta(q) > 1$ , Lemma 3 (i), in Bacry and Muzy [6] yields that

$\sup_{r \in (0,1]} E(\int_0^1 Q_r(u) du)^q < \infty$ . This and (160) show that, with  $k$  as above

$$\sup_{r \in (0,1]} E|Z_r^\kappa(1)|^q \leq \sup_{r \in (0,k^{-1})} E|Z_r^\kappa(1)|^q < \infty \quad (161)$$

which is (63). The last statement of the theorem follows from (63) and the fact that  $\{Z_{1/s}^\kappa\}_{s \geq 1}$  is a martingale.  $\square$

*Proof of Proposition 4.1:* The relation (69) can be shown as in the proof of Proposition 3.1. To see (70), observe that, for  $k > 1/\tau + 1$  and using (69)

$$\begin{aligned} EX_k X_0 &= C_\kappa \int_{k\tau}^{(k+1)\tau} \int_0^\tau EQ_0(u)Q_0(v)|u-v|^{2\kappa-1} dudv \\ &= C_\kappa \int_{k\tau}^{(k+1)\tau} \int_0^\tau |u-v|^{2\kappa-1} dudv \\ &= C_\kappa \tau^{2\kappa+1} \int_k^{k+1} \int_0^1 |u-v|^{2\kappa-1} dudv \\ &= \frac{C_\kappa \tau^{2\kappa+1}}{2\kappa(2\kappa+1)} (|k+1|^{2\kappa+1} + |k-1|^{2\kappa+1} - 2|k|^{2\kappa+1}). \end{aligned} \quad (162)$$

A proof of the relation (71) depends on various cases involving  $s$  and  $t$ . For example, consider the case when  $s > 1$  and  $t - s > 1$ . Then

$$\begin{aligned} C_\kappa^{-1} E Z_r^\kappa(s) Z_r^\kappa(t) &= \int_0^s \int_0^t EQ_r(u)Q_r(v)|u-v|^{2\kappa-1} dudv \\ &=: J_{1,r} + J_{2,r} + J_{3,r} \end{aligned} \quad (163)$$

where  $J_{1,r}$ ,  $J_{2,r}$ , and  $J_{3,r}$  are the integrals restricted to  $|u-v| > 1$ ,  $r < |u-v| < 1$ , and  $|u-v| < r$ , respectively. Observe that

$$J_{1,r} = \int_0^s \int_0^t |u-v|^{2\kappa-1} 1_{\{|u-v|>1\}} dudv < \infty.$$

The term  $J_{2,r}$  can be written as

$$\begin{aligned} J_{2,r} &= \int_0^s du \int_{u+r}^{u+1} dv (v-u)^{c\varphi(2)+2\kappa-1} \\ &+ \left\{ \int_r^1 du \int_0^{u-r} dv + \int_1^s du \int_{u-1}^{u-r} dv \right\} (u-v)^{c\varphi(2)+2\kappa-1} \\ &= s \int_r^1 dz z^{c\varphi(2)+2\kappa-1} \\ &+ \left\{ \int_0^1 du \int_r^u dz + \int_1^s du \int_r^1 dz \right\} z^{c\varphi(2)+2\kappa-1} \\ &\sim r^{c\varphi(2)+2\kappa} 2s \int_1^\infty dw w^{c\varphi(2)+2\kappa-1}. \end{aligned} \quad (164)$$

One can similarly show that

$$I_{3,r} \sim r^{c\varphi(2)+2\kappa} 2s e^{-c\varphi(2)} \int_0^1 dw e^{c\varphi(2)w} w^{2\kappa-1}$$

and deal with the other cases of  $s$  and  $t$ .  $\square$

*Proof of Proposition 4.2:* The result is immediate from the proof of Proposition 3.1.  $\square$

*Proof of Theorem 4.2:* Consider first the case  $t = b^n$ ,  $n \geq 1$ , with  $r < b^n < b^{n-1} < \dots < b < 1$ . Set

$$Q_{r/b^n}^{(n)}(u) = Q_r^{b^n}(b^n u) \quad (165)$$

and

$$Z_r^{\kappa,n}(t) = \int_0^t Q_{r/b^n}^{(n)}(u) dB^\kappa(u). \quad (166)$$

Observe that, by conditioning on  $Q_r(u)$ , using (31) and (14), and making a change of variables  $u \leftrightarrow bu, v \leftrightarrow bv$

$$\begin{aligned} E|Z_r^{\kappa,n}(b^n)|^q &= C_{\kappa,q} E \left( \int_0^{b^n} \int_0^{b^n} Q_r(u)Q_r(v)|u-v|^{2\kappa-1} dudv \right)^{\frac{q}{2}} \\ &= C_{\kappa,q} b^{(\kappa+\frac{1}{2})q} E \left( \int_0^{b^{n-1}} \int_0^{b^{n-1}} Q_b(bu)Q_b(bv)|u-v|^{2\kappa-1} dudv \right)^{\frac{q}{2}} \end{aligned} \quad (167)$$

and similarly

$$\begin{aligned} E|Z_r^{\kappa,1}(b^{n-1})|^q &= C_{\kappa,q} E \left( \int_0^{b^{n-1}} \int_0^{b^{n-1}} Q_r^b(bu)Q_r^b(bv)|u-v|^{2\kappa-1} dudv \right)^{\frac{q}{2}}. \end{aligned} \quad (168)$$

Then, by using the inequality (69) in Chainais *et al.* [9] and independence of  $\{Q_b(u)\}$  and  $\{Q_r^b(u)\}$

$$\begin{aligned} & \left| E|Z_r^{\kappa,n}(b^n)|^q - b^{(\kappa+\frac{1}{2})q} E Q_b(0)^q E|Z_r^{\kappa,1}(b^{n-1})|^q \right| \\ & \leq C_{\kappa,q} b^{(\kappa+\frac{1}{2})q} E \left| \left( \int_0^{b^{n-1}} \int_0^{b^{n-1}} Q_b(bu)Q_b(bv) \right. \right. \\ & \quad \left. \left. Q_r^b(bu)Q_r^b(bv)|u-v|^{2\kappa-1} dudv \right)^{\frac{q}{2}} \right. \\ & \quad \left. - Q_b(0)^q \left( \int_0^{b^{n-1}} \int_0^{b^{n-1}} Q_r^b(bu)Q_r^b(bv)|u-v|^{2\kappa-1} dudv \right)^{\frac{q}{2}} \right| \\ & \leq E \sup_{0 \leq u,v \leq b^n} |Q_b(u)^{\frac{q}{2}} Q_b(v)^{\frac{q}{2}} - Q_b(0)^q| E|Z_r^{\kappa,1}(b^{n-1})|^q \end{aligned} \quad (169)$$

and this yields

$$E|Z_r^{\kappa,n}(b^n)|^q = b^{(\kappa+\frac{1}{2})q} E Q_b(0)^q E|Z_r^{\kappa,1}(b^{n-1})|^q (1 + \epsilon_{r,b,n}^{(1)}) \quad (170)$$

where

$$\begin{aligned} & |\epsilon_{r,b,n}^{(1)}| \\ & \leq \left( E Q_b(0)^q \right)^{-1} E \sup_{0 \leq u,v \leq b^n} \left| Q_b(u)^{\frac{q}{2}} Q_b(v)^{\frac{q}{2}} - Q_b(0)^q \right| \\ & = \tau_{b,q}^{(1)}(b^{n-1}). \end{aligned} \quad (171)$$

Focusing now on the term  $E|Z_r^{\kappa,1}(b^{n-1})|^q$ , we have as above

$$E|Z_r^{\kappa,1}(b^{n-1})|^q = C_{\kappa,q} b^{(\kappa+\frac{1}{2})q} E \left( \int_0^{b^{n-2}} \int_0^{b^{n-2}} dudv \right. \\ \left. Q_{b^2}^b(b^2u)Q_{b^2}^b(b^2v)Q_r^{b^2}(b^2u)Q_r^{b^2}(b^2v)|u-v|^{2\kappa-1} \right)^{\frac{q}{2}} \quad (172)$$

$$E|Z_r^{\kappa,2}(b^{n-2})|^q = C_{\kappa,q} E \left( \int_0^{b^{n-2}} \int_0^{b^{n-2}} dudv \right. \\ \left. Q_r^{b^2}(b^2u)Q_r^{b^2}(b^2v)|u-v|^{2\kappa-1} \right)^{\frac{q}{2}} \quad (173)$$

and

$$E|Z_r^{\kappa,1}(b^{n-1})|^q = b^{(\kappa+\frac{1}{2})q} EQ_{b^2}^b(0)^q E|Z_r^{\kappa,2}(b^{n-2})|^q (1 + \epsilon_{r,b,n}^{(2)}) \quad (174)$$

where

$$|\epsilon_{r,b,n}^{(2)}| \leq \left( EQ_{b^2}^b(0)^q \right)^{-1} E \sup_{0 \leq u,v \leq b^n} \left| Q_{b^2}^b(u)^{\frac{q}{2}} Q_{b^2}^b(v)^{\frac{q}{2}} - Q_b(0)^q \right| \\ = \tau_{b,q}^{(2)}(b^{n-2}). \quad (175)$$

By continuing this way for  $E|Z_r^{\kappa,k}(b^{n-k})|^q$ ,  $k = 2, 3, \dots$ ,  $n$ , and gathering (170), (174), and similar results for  $E|Z_r^{\kappa,k}(b^{n-k})|^q$ , we obtain that

$$E|Z_r^{\kappa}(b^n)|^q = (b^n)^{(\kappa+\frac{1}{2})q} \times \\ EQ_b(0)^q EQ_{b^2}^b(0)^q \dots EQ_{b^{n-1}}^{b^{n-1}}(0)^q E|Z_r^{\kappa,n}(1)|^q \prod_{k=1}^n (1 + \epsilon_{r,b,n}^{(k)}) \\ = (b^n)^{(\kappa+\frac{1}{2})q} EQ_{b^n}(0)^q E|Z_r^{\kappa,n}(1)|^q \prod_{k=1}^n (1 + \epsilon_{r,b,n}^{(k)}) \quad (176)$$

with

$$|\epsilon_{r,b,n}^{(k)}| \leq \tau_{b,q}^{(k)}(b^{n-k}).$$

In the case (21), by (25),  $Q_r^{b^n}(b^n u)$  has the same finite-dimensional distributions as  $Q_{r/b^n}(u)$  and, hence,  $E|Z_r^{\kappa,n}(1)|^q = E|Z_{r/b^n}^{\kappa}(1)|^q$ . Moreover,  $EQ_{b^n}(0)^q = (b^n)^{c\varphi(q)}$  using (102). Together with (176), this yields

$$E|\tilde{Z}_r^{\kappa}(b^n)|^q = (b^n)^{\zeta(q)} E|\tilde{Z}_{r/b^n}^{\kappa}(1)|^q \prod_{k=1}^n (1 + \epsilon_{r,b,n}^{(k)}).$$

The relation (82) for  $t = b^n$  follows by letting  $r \rightarrow 0$  and using the assumptions (80) and (81). This relation for arbitrary  $t \in (0, 1]$  follows from that for  $t = b^n$  by monotonicity.  $\square$

ACKNOWLEDGMENT

The authors would like to thank the anonymous referees for many useful comments and suggestions. V. Pipiras is grateful to ENS Laboratoire de Physique, Lyon, France, for their hospitality.

REFERENCES

[1] R. Riedi, "Multifractals processes," in *Theory and Applications of Long Range Dependence*, P. Doukhan, G. Oppenheimer, and M. S. Taqqu, Eds. Boston, MA: Birkhäuser, 2003, pp. 625–716.

[2] B. B. Mandelbrot, "Intermittent turbulence in self-similar cascades; divergence of high moments and dimension of the carrier," *J. Fluid Mech.*, vol. 62, pp. 331–358, 1974.

[3] J.-P. Kahane and J. Peyriere, "Sur certaines martingales de Benoit Mandelbrot," *Adv. Math.*, vol. 22, pp. 131–145, 1976.

[4] J. Barral and B. B. Mandelbrot, "Multiplicative products of cylindrical pulses," *Probability Theory and Related Fields*, vol. 124, pp. 409–430, 2002.

[5] J. Muzy and E. Bacry, "Multifractal stationary random measures and multifractal random walks with log-infinitely divisible scaling laws," *Phys. Rev. E*, vol. 66, p. 056121, 2002.

[6] E. Bacry and J. Muzy, "Log-infinitely divisible multifractal processes," *Commun. Math. Phys.*, vol. 236, pp. 449–475, 2003.

[7] F. Schmitt and D. Marsan, "Stochastic equations generating continuous multiplicative cascades," *Europ. Phys. J. B*, vol. 20, pp. 3–6, 2001.

[8] F. Schmitt, "A causal multifractal stochastic equation and its statistical properties," *Europ. Phys. J. B*, vol. 34, pp. 85–98, 2003.

[9] P. Chainais, R. Riedi, and P. Abry, "On non-scale-invariant infinitely divisible cascades," *IEEE Trans. Inf. Theory*, vol. 51, no. 3, pp. 1063–1083, Mar. 2005.

[10] P. Chainais, R. Riedi, and P. Abry, "Warped infinitely divisible cascades: Beyond scale invariance," *Traitement du Signal*, vol. 22, no. 1, pp. 27–40, 2005.

[11] P. Mannersalo, I. Norros, and R. H. Riedi, "Multifractal products of stochastic processes: Construction and some basic properties," *Adv. Appl. Probab.*, vol. 34, no. 4, pp. 888–903, 2002.

[12] D. Schertzer and S. Lovejoy, "Physical modeling and analysis of rain and clouds by anisotropic scaling of multiplicative processes," *J. Geophys. Res. D*, vol. 92, no. 8, pp. 9693–9714, 1987.

[13] B. Mandelbrot, "A multifractal walk down Wall Street," *Scientif. Amer.*, vol. 280, no. 2, pp. 70–73, 1999.

[14] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance*. New York, London: Chapman and Hall, 1994.

[15] P. Embrechts and M. Maejima, *Selfsimilar Processes*. San Diego, CA: Academic, 2002.

[16] P. Carmona, L. Coutin, and G. Montseny, "Stochastic integration with respect to fractional Brownian motion," *Ann. l'Inst. Henri Poincaré*, vol. 39, no. 1, pp. 27–68, 2003.

[17] L. Decreusefond and A.S. Üstünel, "Stochastic analysis of the fractional Brownian motion," *Potential Anal.*, vol. 10, pp. 177–214, 1999.

[18] G. Gripenberg and I. Norros, "On the prediction of fractional Brownian motion," *J. Appl. Probab.*, vol. 33, pp. 400–410, 1996.

[19] V. Pipiras and M. S. Taqqu, "Integration questions related to fractional Brownian motion," *Probab. Theory and Related Fields*, vol. 118, no. 2, pp. 251–291, 2000.

[20] E. Bacry, J. Delour, and J. Muzy, "Multifractal random walk," *Phys. Rev. E*, vol. 64, p. 026103, 2001.

[21] C. Ludeña, "L<sup>p</sup>-variations for multifractal fractional random walks," *Ann. Appl. Probab.*, vol. 18, no. 3, pp. 1138–1163, 2008.

[22] V. Pipiras and M. S. Taqqu, "Fractional calculus and its connections to fractional Brownian motion," in *Theory and Applications of Long Range Dependence*, P. Doukhan, G. Oppenheim, and M. S. Taqqu, Eds. Boston, MA: Birkhäuser, 2003, pp. 165–201.

[23] J.-M. Bardet, G. Lang, G. Oppenheim, A. Philippe, and M. S. Taqqu, "Generators of long-range dependent processes: A survey," in *Long-Range Dependence: Theory and Applications*, P. Doukhan, G. Oppenheim, and M. S. Taqqu, Eds. Boston, MA: Birkhäuser, 2003, pp. 579–623.

[24] B. Lashermes, P. Abry, and P. Chainais, "New insights into the estimation of scaling exponents," *Int. J. Wavelets, Multiresol. Inf. Process.*, vol. 2, no. 4, pp. 497–523, 2004.

[25] S. Jaffard, "Wavelet techniques in multifractal analysis," in *Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot*, M. Lapidus and M. van Frankenhuysen, Eds. Providence, RI: Amer. Math. Soc., 2004, vol. 72, no. 2, Proceedings of Symposia in Pure Mathematics, pp. 91–152.

[26] S. Jaffard, B. Lashermes, and P. Abry, "Wavelet leaders in multifractal analysis," in *Wavelet Analysis and Applications*, T. Qian, M. I. Vai, and X. Yuesheng, Eds. Basel, Switzerland: Birkhäuser-Verlag, 2006, pp. 219–264.

[27] S. Kwapien and N. A. Woyczyński, *Random Series and Stochastic Integrals: Single and Multiple*. Boston, MA: Birkhäuser, 1992.

- [28] P. Billingsley, *Convergence of Probability Measures*. New York: Wiley, 1968.
- [29] R. J. Adler, *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*. Hayward, CA: Inst. Math. Statist., 1990, vol. 12, Institute of Mathematical Statistics Lecture Notes -Monograph Series.

**Patrice Abry** (M'05–SM'07) was born in Bourg-en-Bresse, France, in 1966. He received the degree of Professeur-Agrégé de Sciences Physiques, in 1989 at Ecole Normale Supérieure de Cachan, France, and completed work on the Ph.D. degree in physics and signal processing, at Ecole Normale Supérieure de Lyon and Université Claude-Bernard Lyon I, France, in 1994.

Since October 1995, he has been a permanent CNRS Researcher at the Laboratoire de Physique of Ecole Normale Supérieure de Lyon. His current research interests include wavelet-based analysis and modeling of scaling phenomena and related topics (self-similarity, stable processes, multi-fractal,  $1/f$  processes, long-range dependence, local regularity of processes, infinitely visible cascades, departures from exact scale invariance, etc.). Hydrodynamic turbulence and the analysis and modeling of computer network tele-traffic are the main applications under current investigation. He is also involved in the study of baroreflex sensitivity with a French medical group at Hospices Civils de Lyon.

Dr. Abry received the AFCET-MESR-CNRS prize for best Ph.D. dissertation in signal processing for the years 1993–1994 and is the author of a book *On-delettes et Turbulences—Multirésolution, algorithmes de décompositions, invariance d'échelle et signaux de pression*, published in October 97, by Diderot, Paris, France. He also is the Co-Editor of *Scaling, Fractals and Wavelets*, Iste, Wiley, 2009.

**Pierre Chainais** (M'05) received the degree of Agrégation de Sciences Physiques in 1996. He received the B.S. degree in mathematics in 1995, the M.Sc. degree in physics in 1997, and the Ph.D. degree in physics in 2001 from the Ecole Normale Supérieure de Lyon (France). He joined Clermont-Ferrand University as a Visiting Research Associate in Signal Processing in 2002. He is currently an Assistant Professor of Physics and Signal Processing at Clermont-Ferrand University. His research interests lie in the areas of self-similar stochastic processes, multifractal analysis, wavelets, and scaling laws with applications to physical systems.

**Laure Coutin** received the Ph.D. degree in mathematics from the University of Orléans, Orléans, France, in 1994.

She has been an Associate Professor at the University of Toulouse 3 (France) from 1995 to 2006. She has been a Professor at the University Paris Descartes since February 2006. Her current research interests include Gaussian and Lévy processes, and rough paths theory.

**Vladas Pipiras** received the B.A. degree in mathematics from Vilnius University, Vilnius, Lithuania, in 1996, the D.E.A. degree in probability and applications from Paris 6 University, France, in 1997, and the Ph.D. degree in mathematics from Boston University, Boston, MA, in 2002.

In 2002, he joined the Department of Statistics and Operations Research, University of North Carolina, Chapel Hill, where he is now an Associate Professor. His research interests include long-range dependence, self-similarity, multifractals, heavy tails, stable distributions, and multiscale (wavelet) methods.