

Des cascades infiniment divisibles multifractales.

Emmanuel Bacry

CMAP, Ecole Polytechnique 91128 Palaiseau Cedex, France

emmanuel.bacry@polytechnique.fr

<http://www.cmap.polytechnique.fr/~bacry>

Jean-François Muzy

Université de Corse, Grossetti, 20250 Corté, France

CRPP, 33600 Pessac, France

Multifractal Processes and Scale Invariance

$\delta_l X(t) = X(t+l) - X(t)$: Increment at time t , scale l .

$$M(q, l) = E(|\delta_l X(t)|^q) \sim K_q l^{\zeta_q}.$$

$\implies$ “Scale invariance” of $\delta_l X(t)$

ζ_q : Multifractal exponents

- Self-similar (Monofractal) Processes

$\zeta_q = qH$ is linear, H : Hurst exponent.

Brownian motion : $H = 1/2$

Levy walk (α -stable process) : $H = 1/\alpha$,

- Multifractal Processes

ζ_q is non linear



$\implies$ Existence of an “integral” scale $l = L$.

$\implies M(q, l) \simeq K_q l^{\zeta_q}, \quad \forall l \leq L \quad (\zeta_q \text{ convex}).$

Multi/Mono fractal Processes

- Self Similar processes ($\delta_l X(t) = X(t+l) - X(t)$)

$$\{\delta_{\lambda l} X(\lambda t)\}_t \stackrel{law}{=} \lambda^H \{\delta_l X(t)\}_t$$

$\implies$ a single exponent H

$\implies \zeta_q = qH$, **monofractal** processes

- **Multifractality** (*Castaing 90*) $\simeq H$ is stochastic

$$\delta_{\lambda l} X(\lambda t) \stackrel{law}{=} W_\lambda \delta_l X(t), \quad (t \leq L)$$

W_λ : log-infinitely divisible, independant of X

G : law of H

- $\zeta_q = \ln \hat{G}(-iq) \implies$ **non-linear**

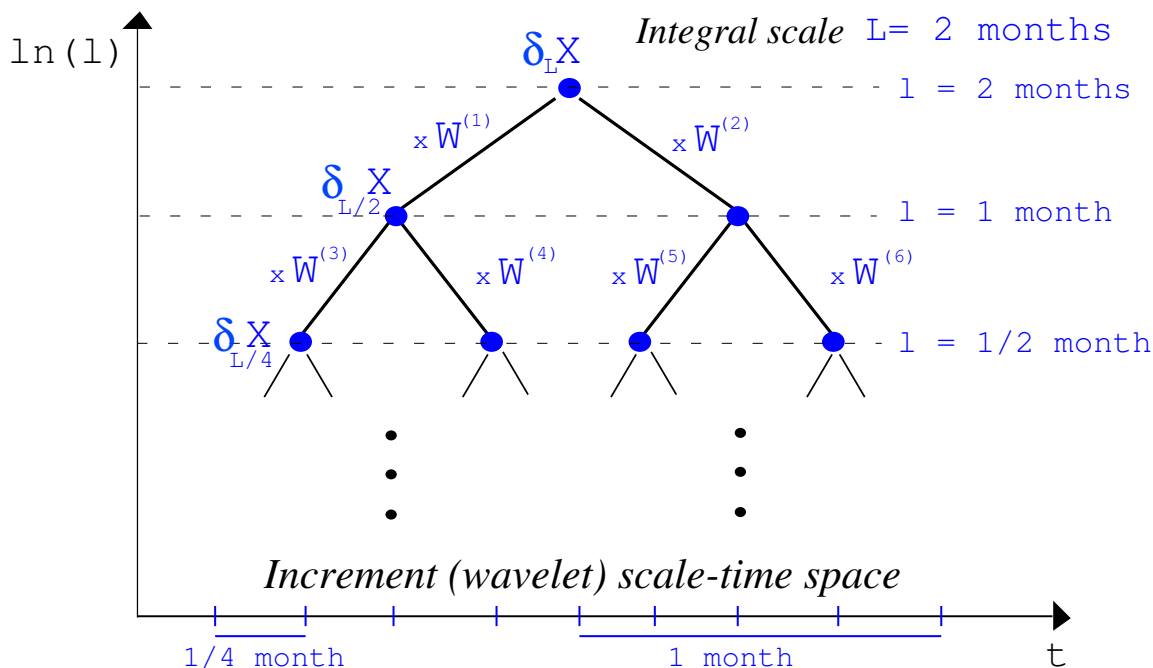
$$\bullet \quad P_{\lambda l}(x) = \int du e^{-u} P_l(e^{-u} x) G^{\ln \lambda}(u)$$

$P_l(x)$: pdf of $\delta_l X(t)$

$\implies$ **Shape of pdf changes across scales.**

Cascade processes

$$\{\delta_{\lambda l} X(\lambda t)\}_t = \lambda^H \{\delta_l X(t)\}_t = W_\lambda \{\delta_l X(t)\}_t$$



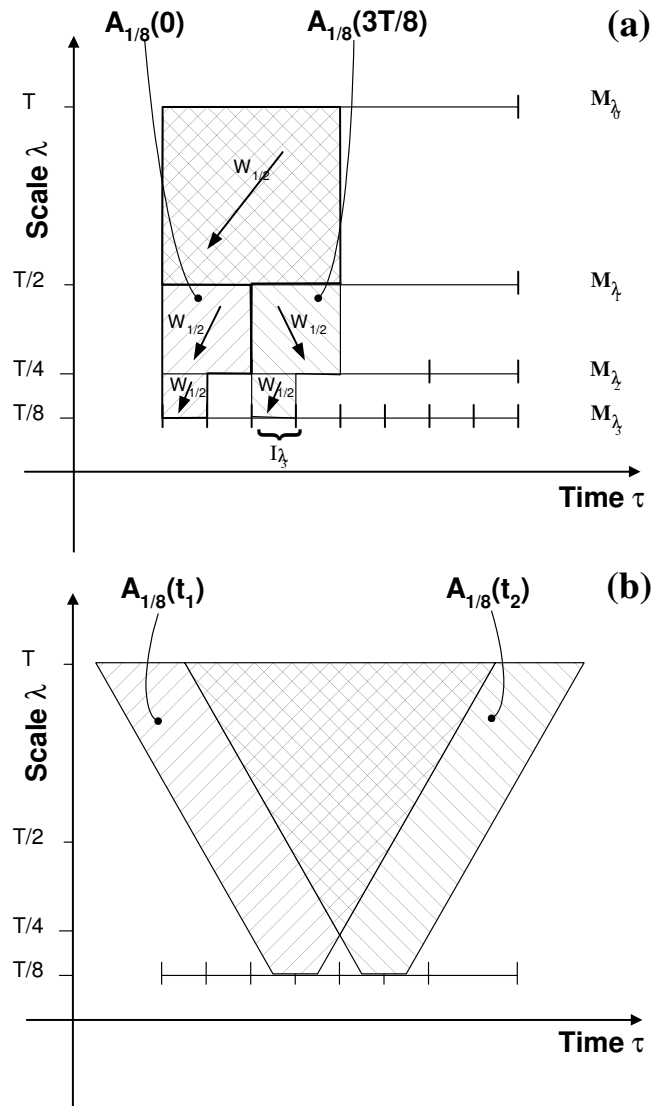
$\Rightarrow$ **Correlation structure**

Magnitude : $\omega_l(t) = \frac{1}{2} \ln |\delta_l X(t)|^2$

$$Cov(\omega_l(t), \omega_l(t + \tau)) \simeq -C \ln(\tau/L), \quad l \ll \tau < L$$

- **Particular scale ratio (discrete scale invariance)**
- **Increments are non stationary**
- **Top-Bottom approach**

Infinitely divisible MRW



- P : infinit. div. stoch. 2D measure distributed with respect to $\mu(dt, dl) = dt dl / l^2$.

$$E\left(e^{pP(\mathcal{A})}\right) = e^{\phi(-ip)\mu(\mathcal{A})},$$

$$(\phi : \text{Lévy Khintchine, } \phi(-i) = 0)$$

- $M_l(I) = \int_I e^{\omega_l(t)} dt$ with $\omega_l(t) = P(\mathcal{A}_l(t))$

- $M(dt) = \lim_{l \rightarrow 0+} M_l(dt)$

Main Results (Measures)

(Bacry, Muzy, 2002)

Existence - Non degeneracy - Moments

- with probability one, $M_l(dt) \xrightarrow{weakly} M(dt)$,
- $\zeta_q = q - \phi(-iq)$
- $\exists \epsilon > 0, \zeta_{1+\epsilon} > 1 \Rightarrow E(M([0, t])) = t.$
- $\zeta_q > 1 \iff E(M([0, t])^q) < +\infty.$

Exact Scaling (*particular cone shape*)

- $\{\omega_{\lambda l}(\lambda t)\}_{t \leq L} =^{law} \Omega_\lambda + \{\omega_l(t)\}_{t \leq L}$,
where Ω_λ is inf. div. with $E(e^{iq\Omega_\lambda}) = \lambda^{-\varphi(q)}.$
- $\{M([0, \lambda t])\}_{t \leq L} =^{law} W_\lambda \{M([0, t])\}_{t \leq L}$,
with $W_\lambda = \lambda e^{\Omega_\lambda}$, for fixed $\lambda \in]0, 1[.$
- $E(M([0, t])^q) = \left(\frac{t}{L}\right)^{\zeta_q} E(M[0, L])^q, \forall t \leq L$

Asymptotic Scaling (*“generic” cone shape*)

- $E(M([0, t])^q) \sim_{t \rightarrow 0+} \left(\frac{t}{L}\right)^{\zeta_q} E(M([0, L])^q).$

Main Results (Processes)

(Bacry, Muzy, 2002)

Definition (using Wiener noise $dW(u)$)

- $X(t) = \lim_{l \rightarrow 0^+} \int_0^t e^{\omega_l(u)/2} dW(u),$

Exact Scaling (*particular cone shape*)

- $\zeta_q > 1$ “ $\iff$ ” $E(|X(t)|^{2q}) < +\infty.$
- $\{X(t)\}_{t \leq L} =^{law} W_\lambda \{X(t)\}_{t \leq L}, \quad \forall \lambda \in]0, 1[,$
- $E(|X(t)|^{2q}) = \left(\frac{t}{L}\right)^{\zeta_q} E(|X(L)|^{2q}), \quad \forall t \leq L.$

Asymptotic Scaling (*“generic” cone shape*)

- $E(|X(t)|^{2q}) \sim_{t \rightarrow 0^+} \left(\frac{t}{L}\right)^{\zeta_q} E(|X(L)|^{2q}).$

Examples

- $P(dt, dl) = 0$
→ $\zeta_q = q$.
→ M is the Lebesgue measure, (X is a Brownian motion)
- $P(dt, dl)$ is normal
→ $\zeta_q = q(1 + \frac{\lambda^2}{2}) - \frac{\lambda^2}{2}q^2$.
→ Exact scaling case : Log normal MRW (*Bacry, Delour, Muzy 2000*),
- $P(dt, dl)$ is (compound) Poisson
→ $\zeta_q = qm - C(E(W^q) - 1)$
→ Asymptotic scaling case : Product of cylindrical pulses (*Barral, Mandelbrot 2000*),
- $P(dt, dl)$ is (left-sided) α stable
→ $\zeta_q = qm - \sigma^\alpha |q|^\alpha$
- $P(dt, dl)$ is Gamma
→ $\zeta_q = qm - C\gamma^2 \ln(\frac{\gamma}{\gamma-q})$

The log-normal MRW model

(Bacry, Delour, Muzy 2000)

- Discrete time : Δt

$$X_{\Delta t}(t) = X_{\Delta t}(t - \Delta t) + e^{\Omega_{\Delta t}(t)} \epsilon_{\Delta t}(t)$$

- $\Omega_{\Delta t}$: Gaussian stationary process

independant of $\epsilon_{\Delta t}(t)$

with autocorrelation $R_{\Delta t}(t)$

- $\epsilon_{\Delta t}$: White Gaussian noise

$$\langle \epsilon_{\Delta t} \rangle = 0, \langle \epsilon_{\Delta t}^2 \rangle = \sigma^2 \Delta t,$$

- $\phi(-i) = 0 \iff \langle \Omega_{\Delta t} \rangle = -Var(\Omega_{\Delta t}) = R_{\Delta t}(0)$
 $\implies Var(X_{\Delta t}(t)) = \sigma^2 t$

- Exact scaling

$$\implies R_{\Delta t}(t) = \begin{cases} \lambda^2 \ln \left(\frac{L}{(|t| + \Delta t)} \right) & \text{for } |t| \leq L \\ 0 & \text{otherwise} \end{cases}$$

The log-normal MRW model

- Continuous scale invariance

$$\begin{aligned}
 & \bullet \langle |X(t)|^q \rangle = K_q (l/L)^{\zeta_q}, \quad \forall t \leq L, \\
 & K_q = L^{q/2} \sigma^q (q-1)!! \prod_{k=0}^{q/2-1} \frac{\Gamma(1-2\lambda^2 k)^2 \Gamma(1-2\lambda^2 (k+1))}{\Gamma(2-2\lambda^2 (q/2+k-1)) \Gamma(1-2\lambda^2)}, \\
 & \bullet \zeta_q = \ln \hat{G}(iq) = q/2 - q(q/2 - 1)\lambda^2
 \end{aligned}$$

- $\langle |X(t)|^q \rangle = K'_q t^q, \quad (t \gg L)$

$\simeq$ Brownian at large scale ($\zeta_q = q/2$)

- $\zeta_q < 1 \quad \text{“} \Longleftrightarrow \text{”} \quad \langle |X|^q \rangle = +\infty$

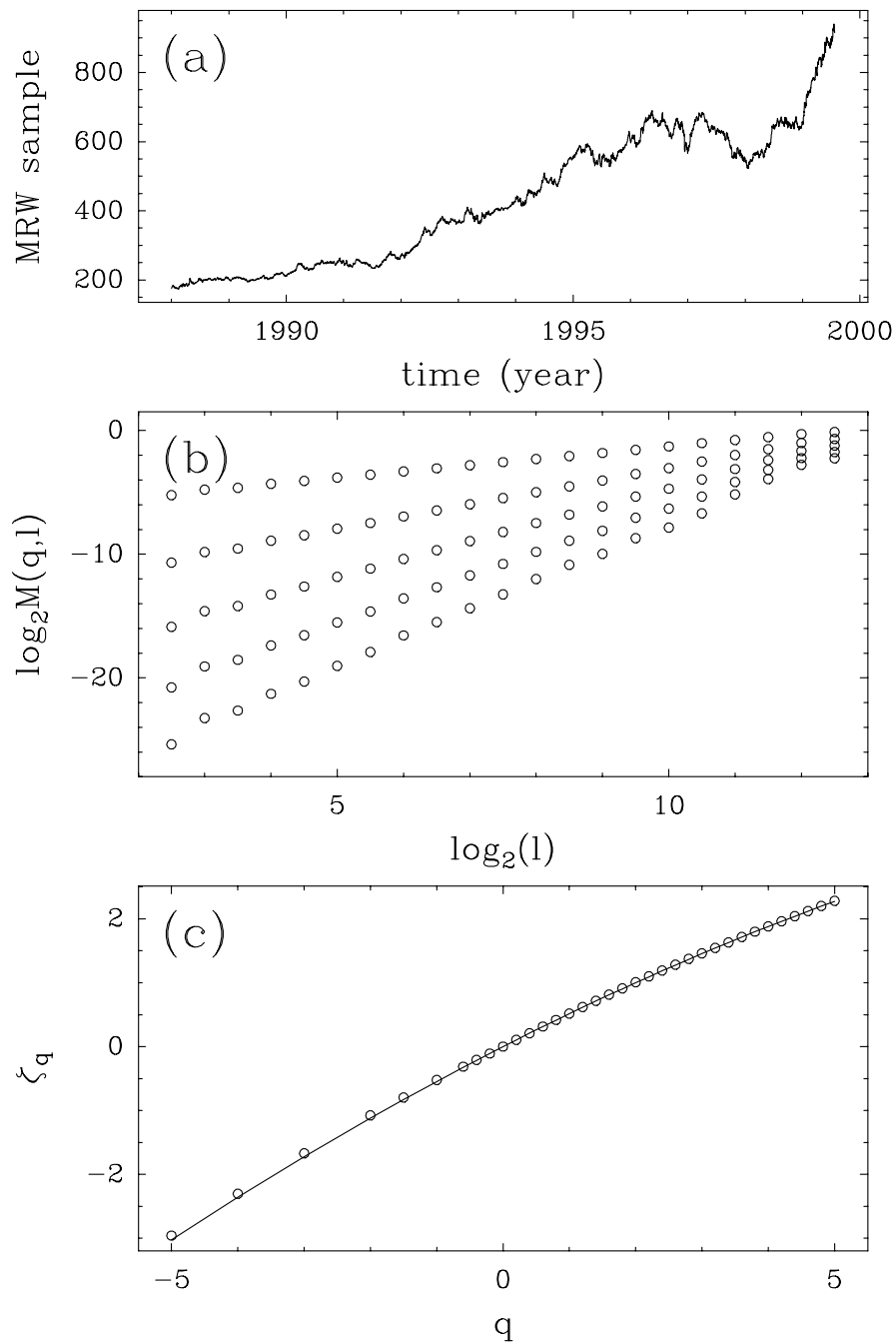
Fat tails

- $\langle |\delta_l X(t)|^{2p} |\delta_l X(t+\tau)|^{2p} \rangle \simeq^{l \ll \tau} C_{2p} \left(\frac{\tau}{L}\right)^{-\lambda^2 p^2},$

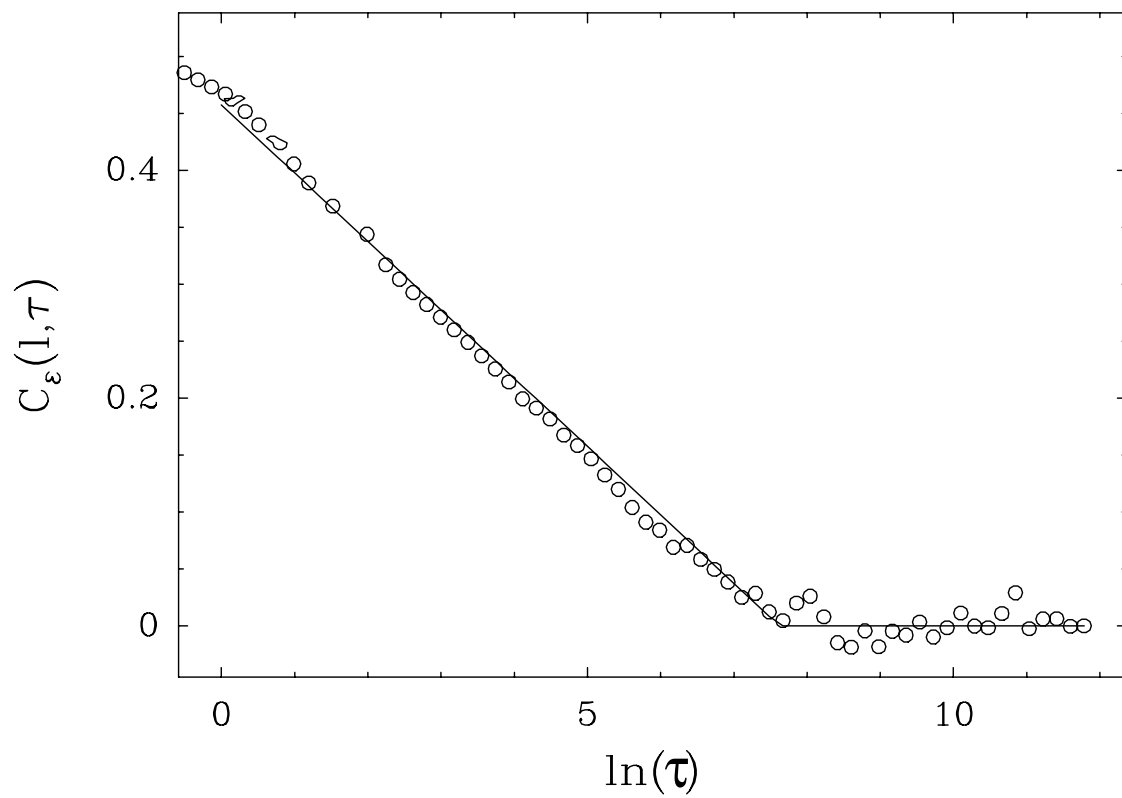
• “Magnitude log correlations” for $\lambda \ll 1$ (2004)
 $Cov(\omega_l(t), \omega_l(t+\tau)) \simeq -\lambda^2 \ln(\tau/L), \quad l < \tau < L$

where $\omega_l(t) = \frac{1}{2} \ln |\delta_l X(t)|^2$

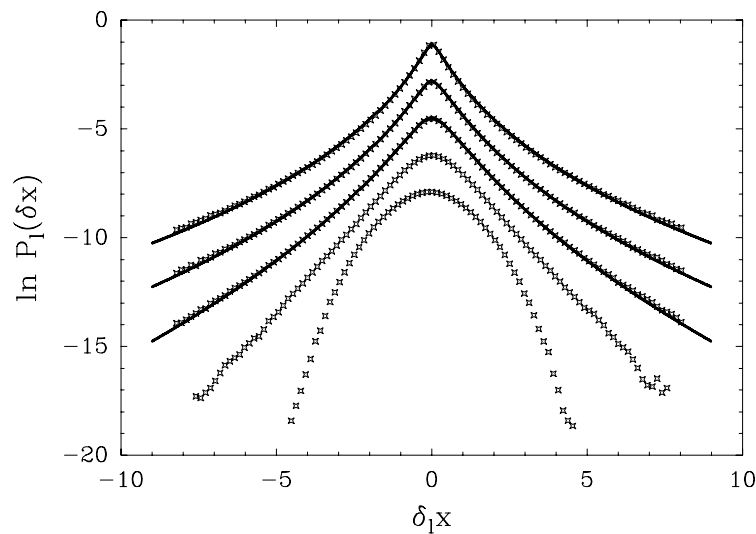
Multifractal analysis of a log-normal MRW



Magnitude correlation of a log-normal MRW

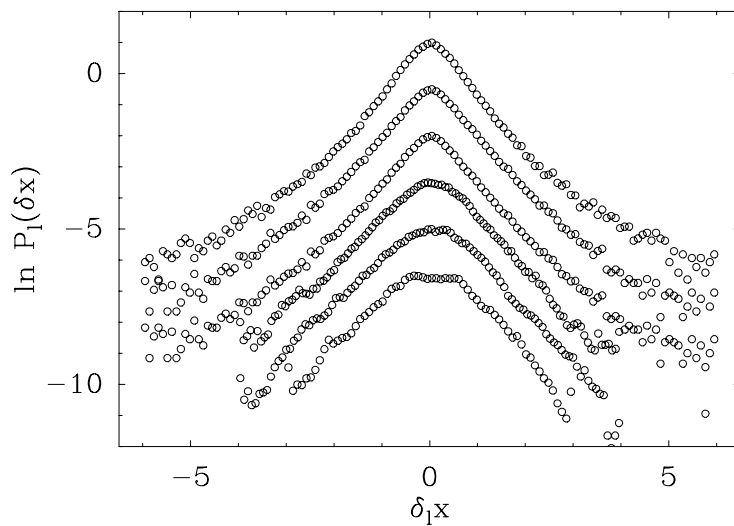


Continuous deformation of increment pdf's across scales



MRW

oooo Numerical estimation
— Castaing prediction



S&P Futures

oooo Numerical estimation

Parameter Estimation

(Bacry, Kojemiak, Muzy, 2004)

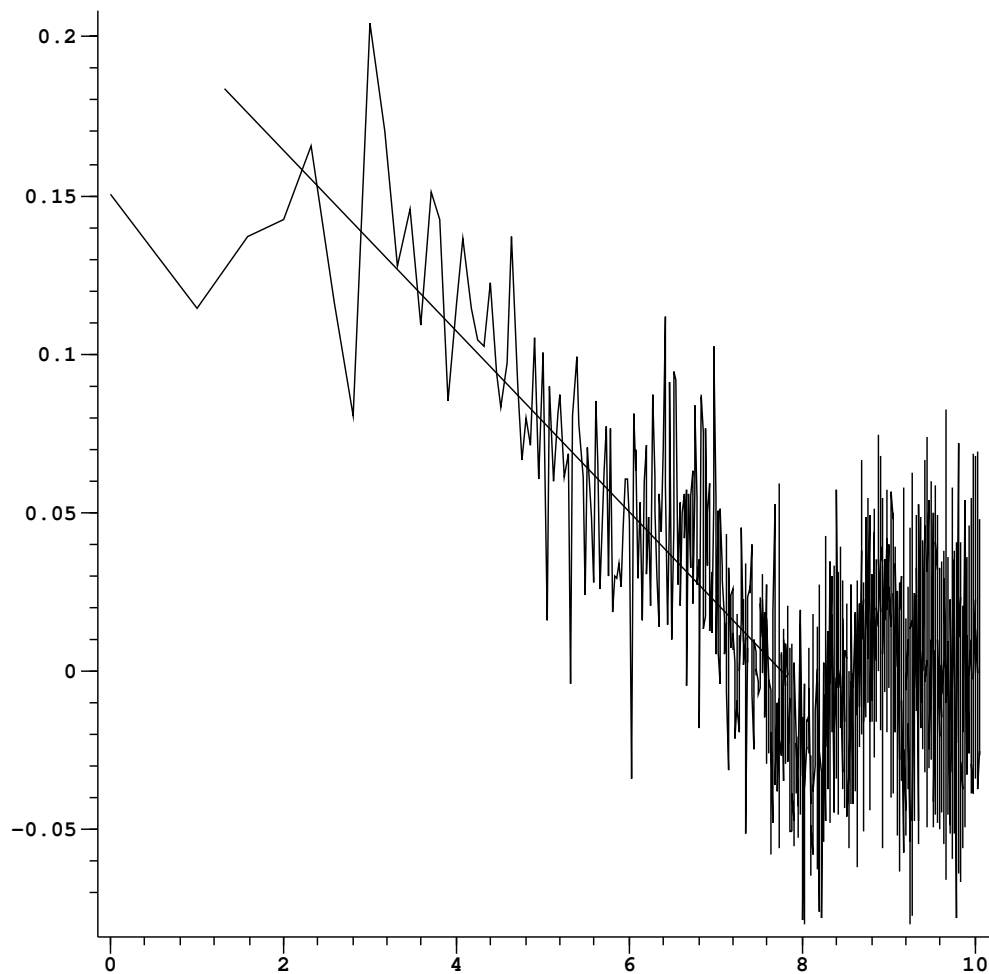
- Integral scale L
→ decorrelation scale
- Variance σ^2
→ variance of the increments
- Multifractal coefficient λ^2
→ correlation of the log of the increments $\omega_l(t)$
$$\text{Cov}(\omega_l(t), \omega_l(t + \tau)) \simeq -\lambda^2 \ln(\tau/L), \quad l < \tau < L$$

⇒ GMM estimation

Parameter Estimation (daily data)

French Index daily data

1/1/1973 to 31/12/97 (6239 points)

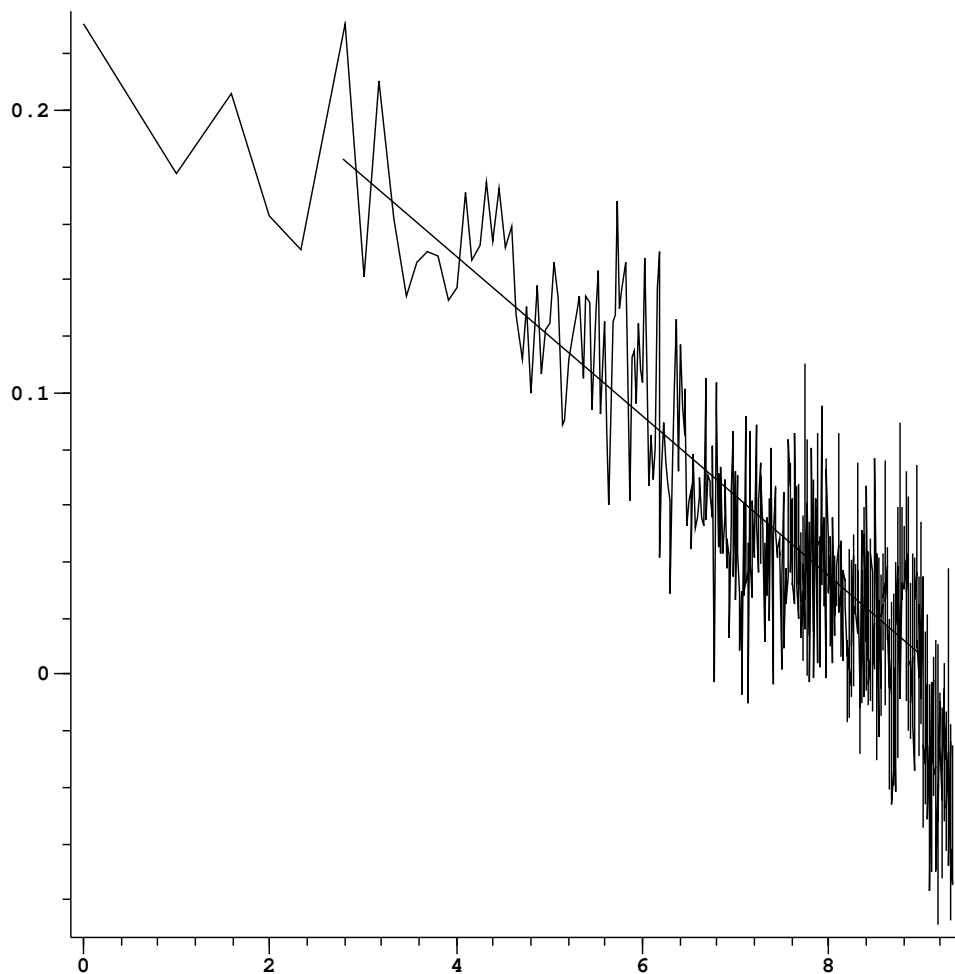


$$\lambda^2 = 0.029 \pm 0.01, L = 1 \text{ year } (\in [6 \text{ months}, 2 \text{ years}])$$

Parameter Estimation (daily data)

Italian Index daily data

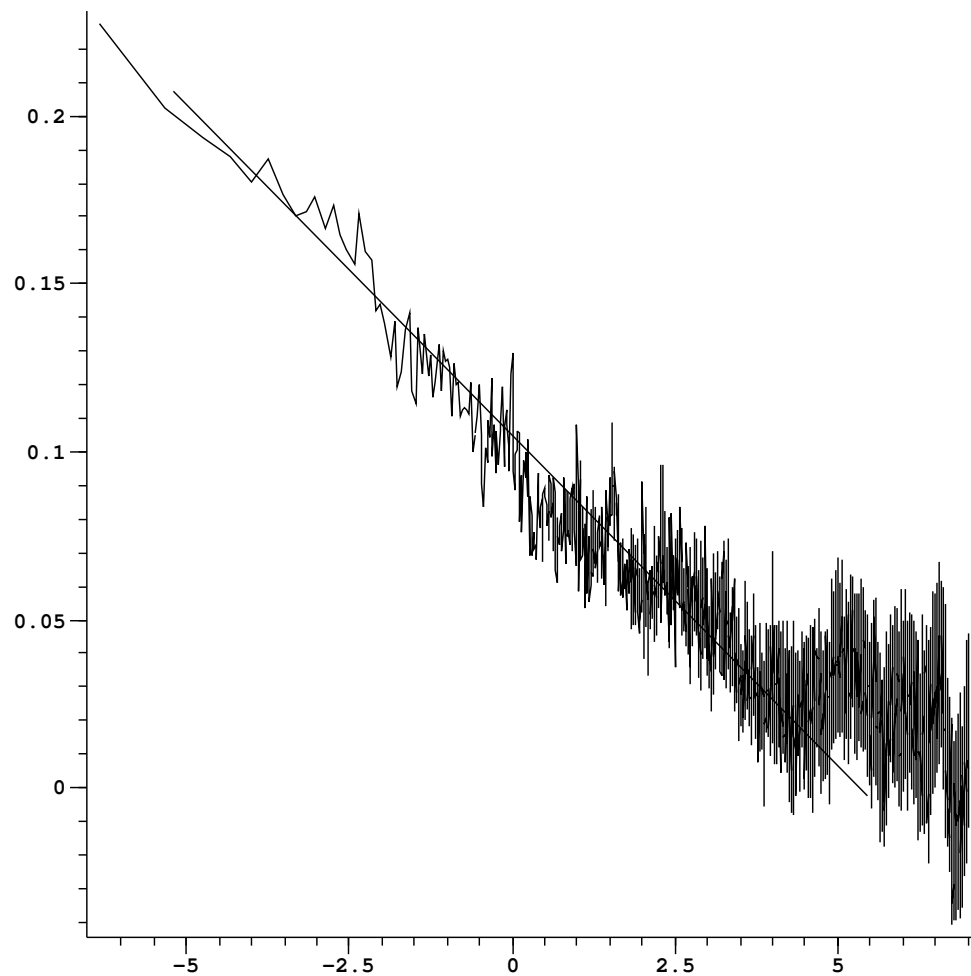
1/1/1973 to 31/12/97 (6239 points)



$$\lambda^2 = 0.029 \pm 0.015, \quad L = 2 \text{ years } (\in [1 \text{ year}, 5 \text{ years}])$$

Parameter Estimation (intraday)

S&P 500 intraday data (5mn ticks, 1996-1998)



$\lambda^2 \simeq 0.03$, $L \simeq 6$ months

Parameter Estimation

Series	Size	λ^2	L
S&P500 index	7.10^4	0.03	6 months
Future S&P500	7.10^4	0.025	3 years
FTSE100 index	7.10^4	0.028	1.2 year
Future FTSE100	7.10^4	0.029	1 year
Future JY/USD	7.10^4	0.02	6 months
Nikkei 225	7.10^4	0.030	1.5 year
Future Nikkei	7.10^4	0.02	6 months
Japanese Yen	7.10^4	0.022	1.0 year
French index	6.10^3	0.029	1 year
Italian index	6.10^3	0.029	2 years
Canadian index	6.10^3	0.032	1.5 years
German index	6.10^3	0.027	6 years
UK index	6.10^3	0.023	6 years
hong-kong index	6.10^3	0.037	6 years

Aggregation properties

- For $\lambda \ll 1$, analytic formula for p -points any order moments, volatility correlation structure across scales...
- ACTUALLY, for $\lambda \ll 1$, at any scale l

$$\delta_l X(nl) \simeq \epsilon_l[n] e^{\Omega_l[n]}$$

- $\epsilon_l[n]$ Gaussian white noise
- $e^{\Omega_l[n]}$ log-normal stochastic variance at scale l

$\Rightarrow$ Variance prediction/estimation

Prospects

(emmanuel.bacry@polytechnique.fr)

Theory

log-normal (inf. div.) parameter estimation

Asymetry past/future (Leverage, HARCH effect,...)

Multivariate MRW (Log Normal)

Multifractal formalism

•

•

Application (mainly finance)

(multivariate) Volatility prediction/estimation

(multivariate) Value at Risk (VaR) prediction/estimation

⇒ non-gaussian dynamical portfolio management.

•

•